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math workshop for children

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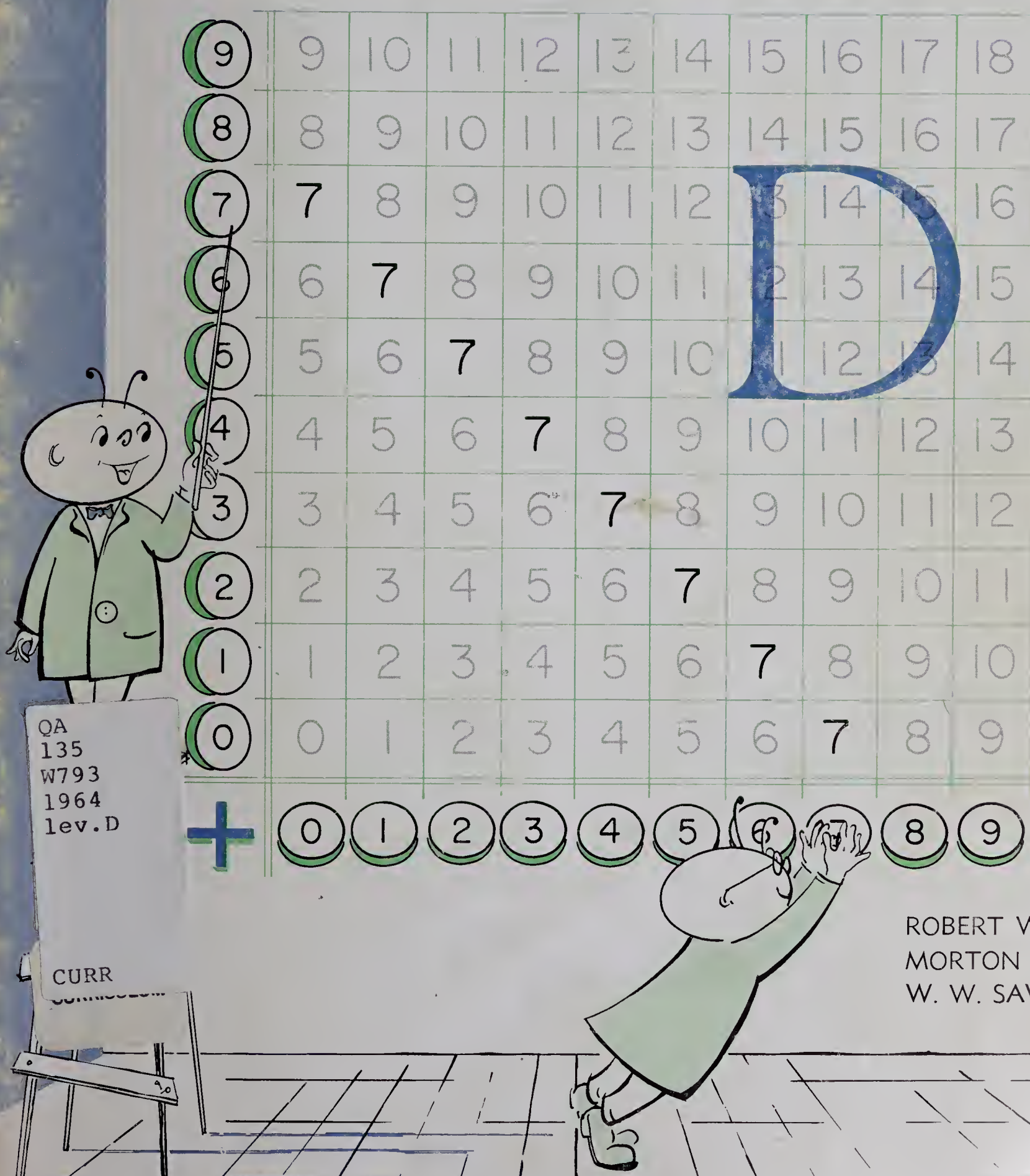

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Numo drawings and Art Direction

Art Staff:

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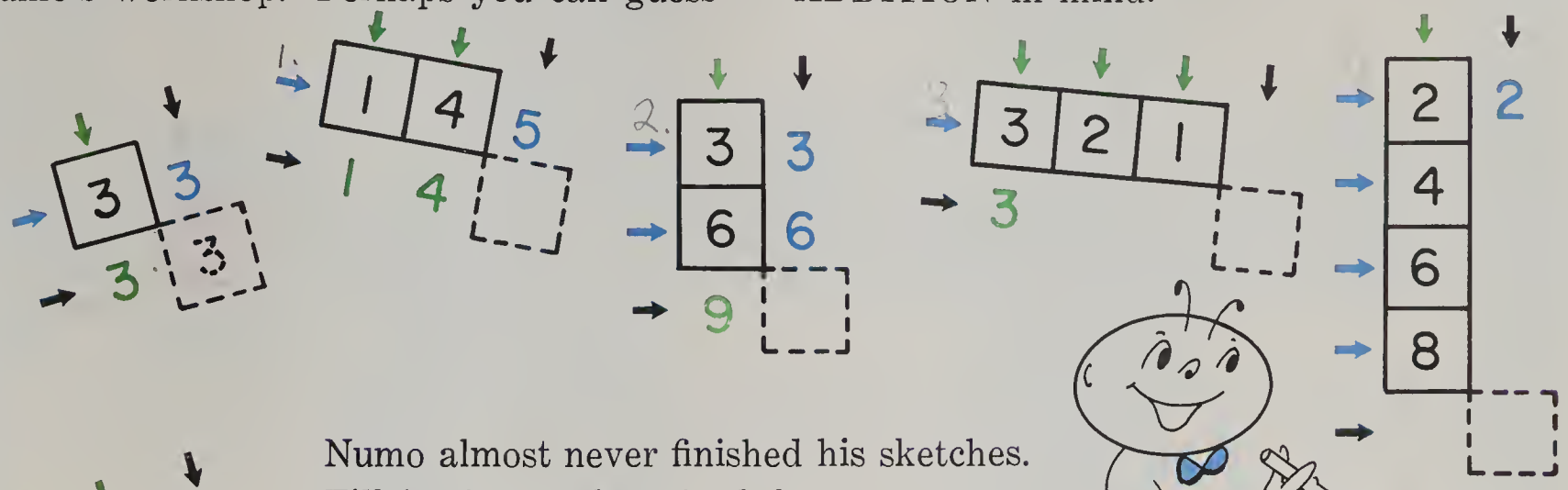
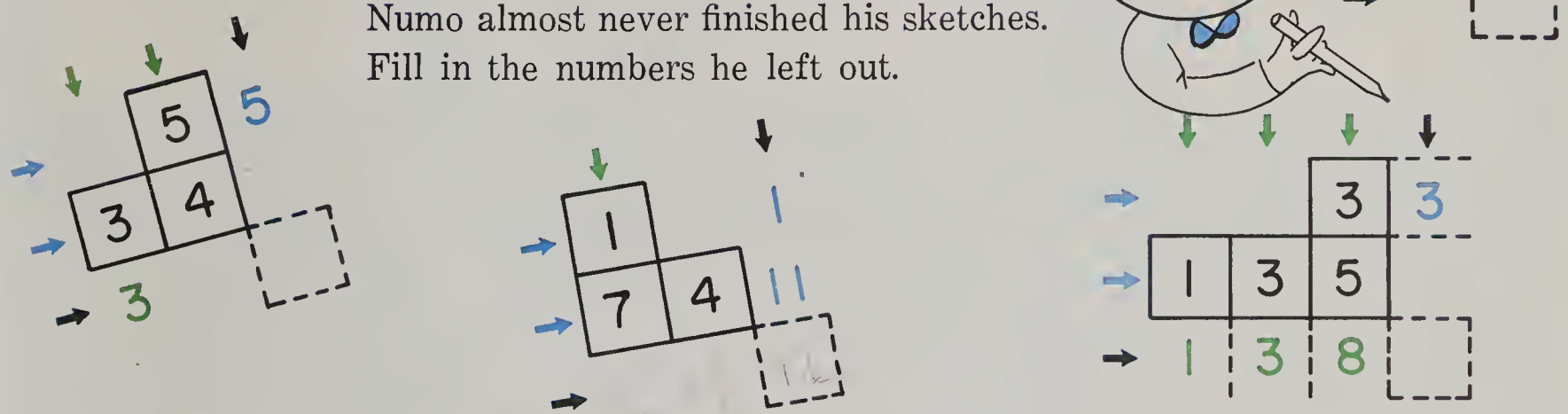
You may not find this fact … your history books, but if Numo didn't invent it, then who did?

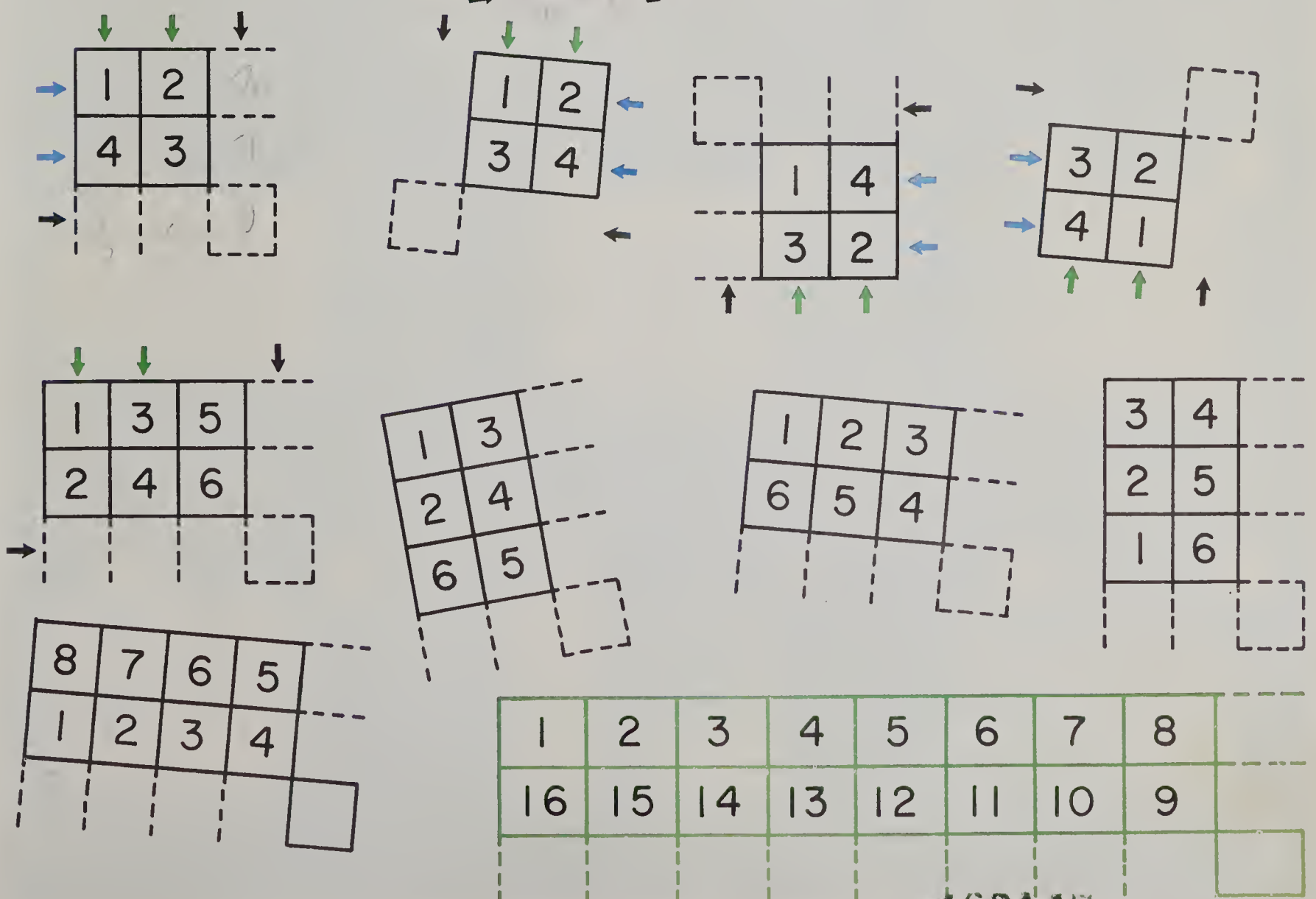
Below are some of the notes we found in Numo's workshop. Perhaps you can guess what went on in his mind as he worked on different varieties of the cross number puzzle.

In these examples he seemed to have ADDITION in mind.



Numo almost never finished his sketches. Fill in the numbers he left out.





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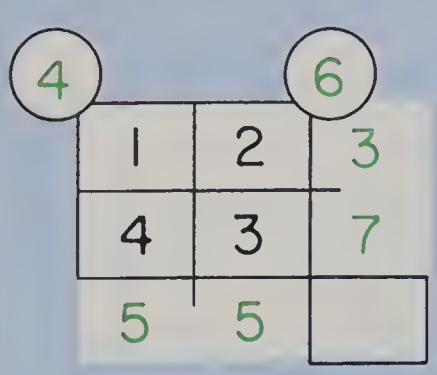
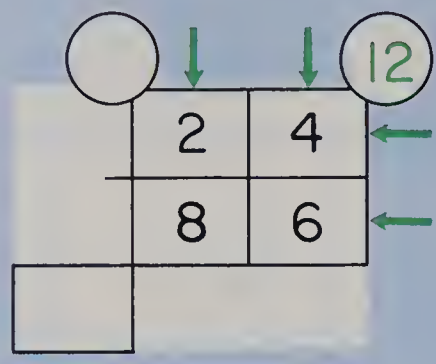
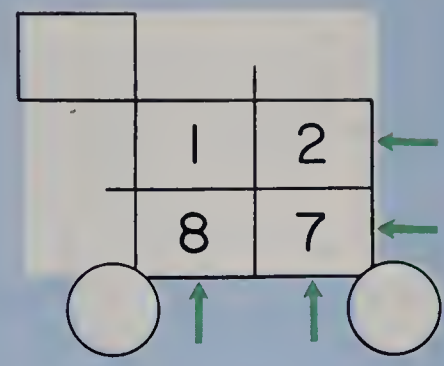
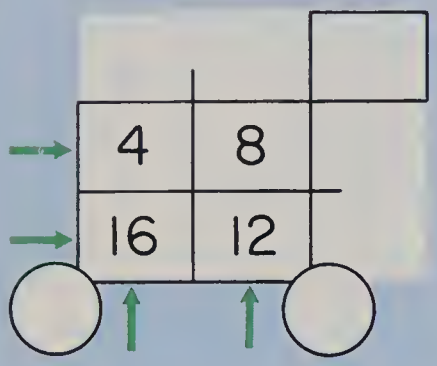
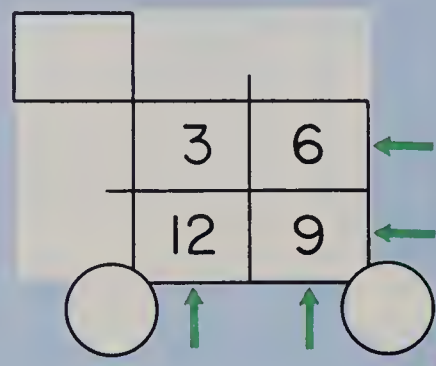
Numo also invented "ears" for his Cross Number Puzzles.
Can you discover the purpose of these ears?

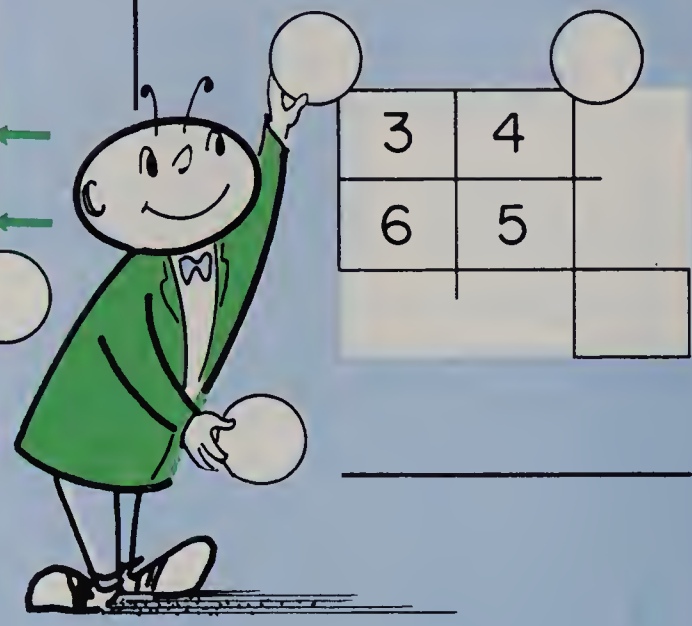




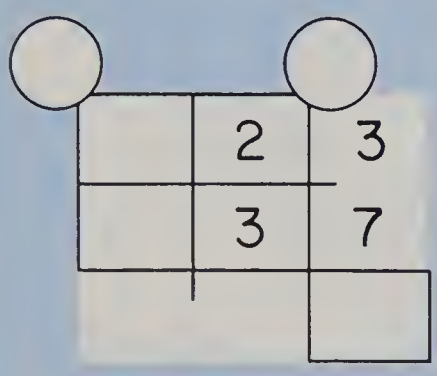


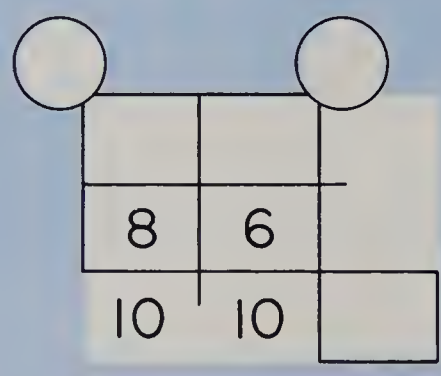


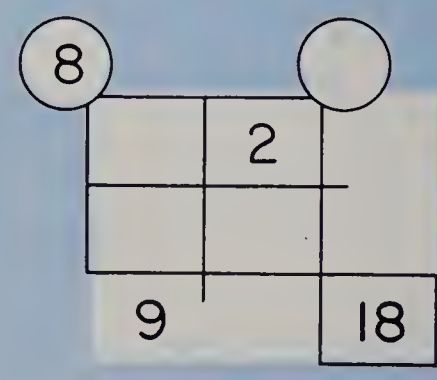


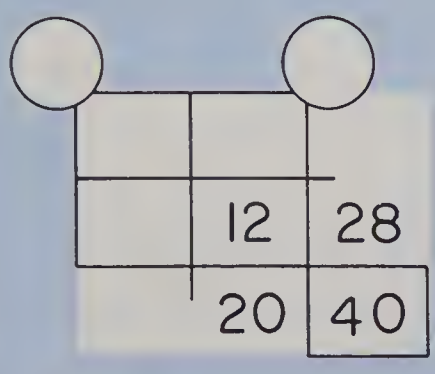


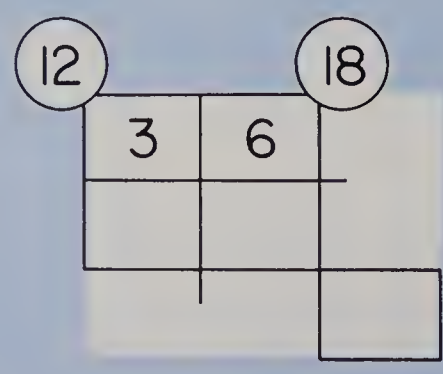
On some pages of his scrap book, Numo experimented a lot. He found that he could begin with numbers in several places. Here are a few of his experiments.

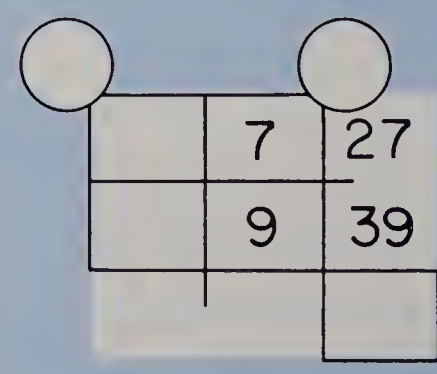




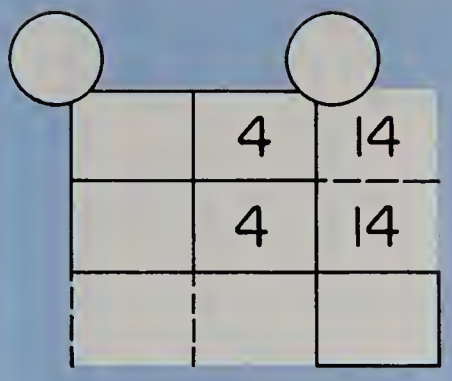


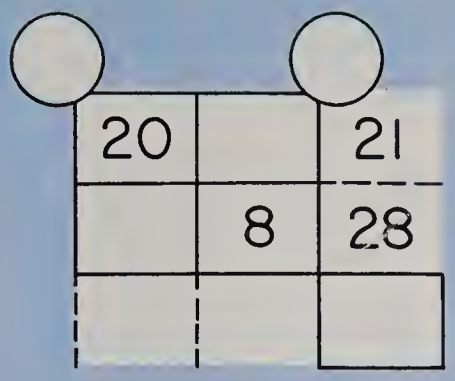


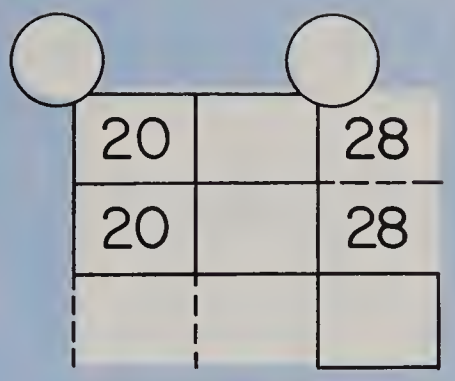


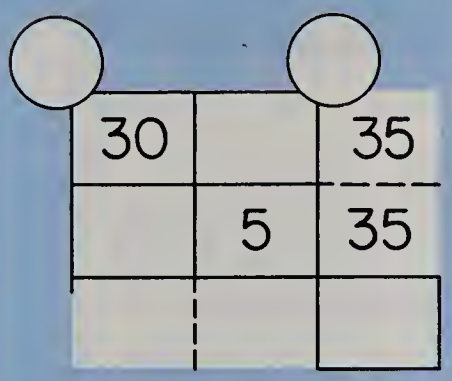


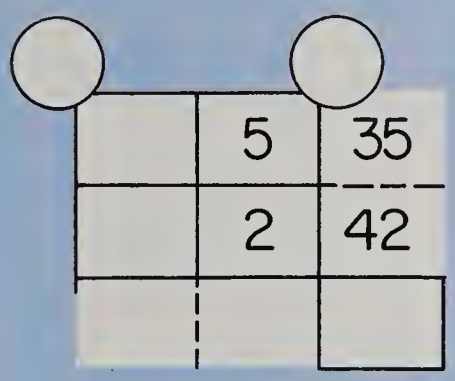
More Experiments.

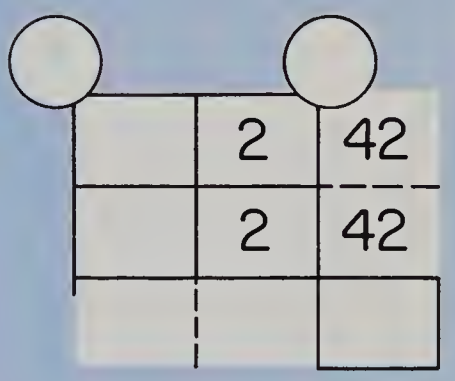




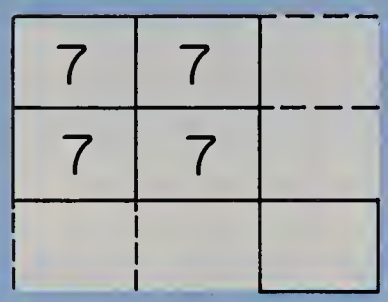


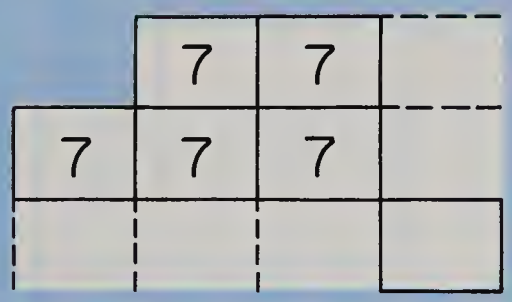


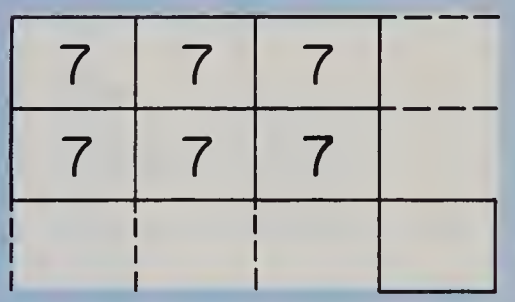


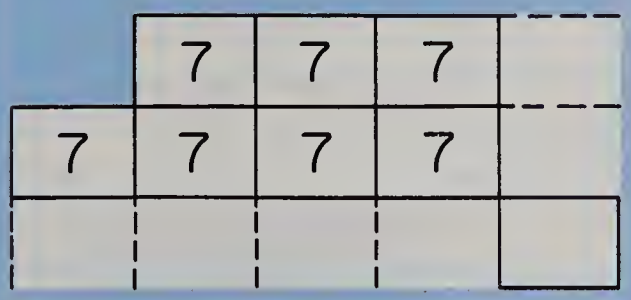


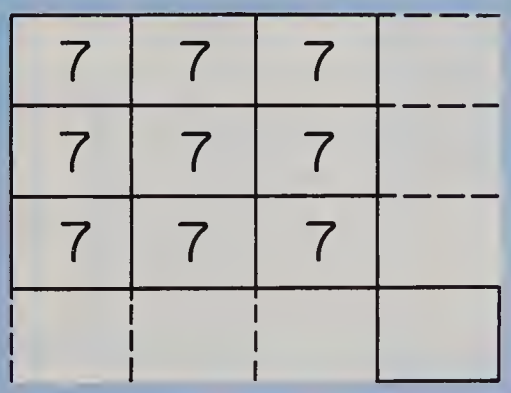
Experiments . . . . without ears!

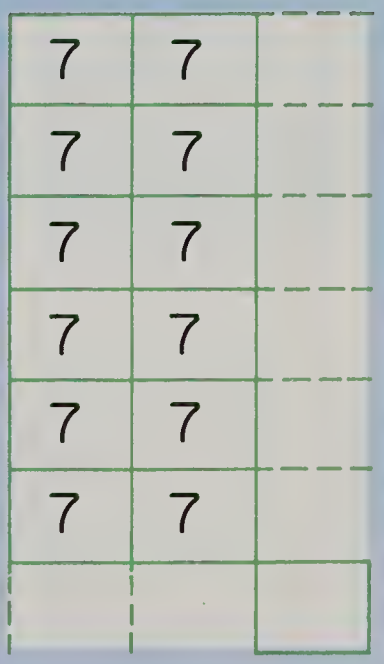


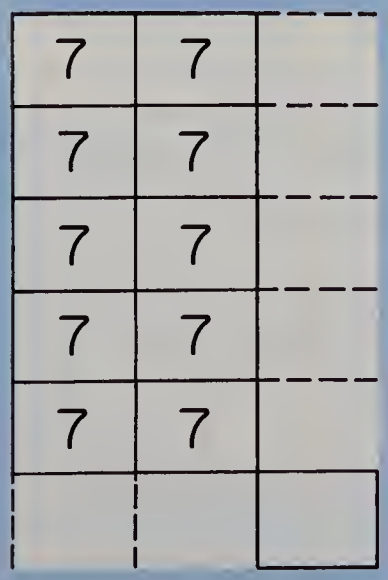


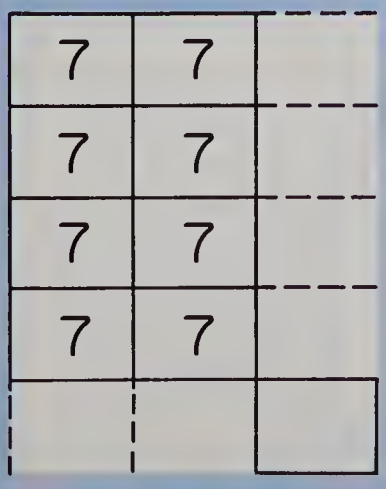


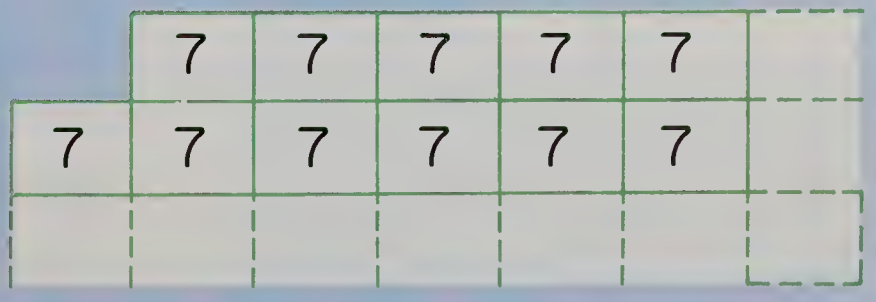












a

10		17		
	5	5		

b

		6		
	20	28		
14				

c

		33		
0		9		
	12			

d

		25	
	0	5	
	20		

e

		15	
	10	15	
10			

f

		15	
	7	7	
10			

g

		18	
10		16	
20			

h

	10	13	
		29	
	30		

i

5		35	
		36	
11			

j

	4	14	
	8	48	

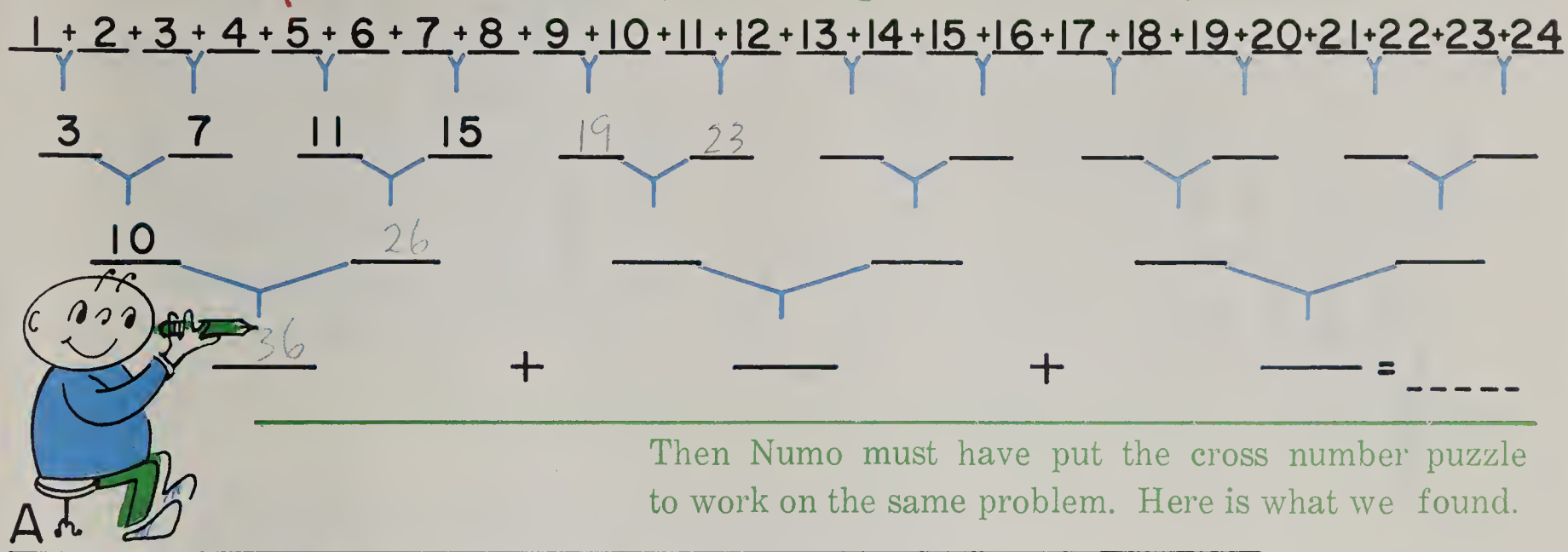
k

10		18	
	8	18	

l

		36	
30		36	
60			

How many is the sum of all the whole numbers 1 through 24? After this question in Numo's notebook, he had begun the work. Can you finish it for him?



A

Then Numo must have put the cross number puzzle to work on the same problem. Here is what we found.

1	2	3	4	5	6	7	8	9	10	11	12	?
								16	15	14	13	?
											25	

B The problem of the green ? was tackled this way—

1	2	3	4	5	6	?
				8	7	?

C The problem of the blue ? was tackled this way—

13	14				18	?
					19	?

D

1	6	
2	5	
3	4	

E

7	12	
8	11	
9	10	

F

13	18	
14	17	
15	16	

G

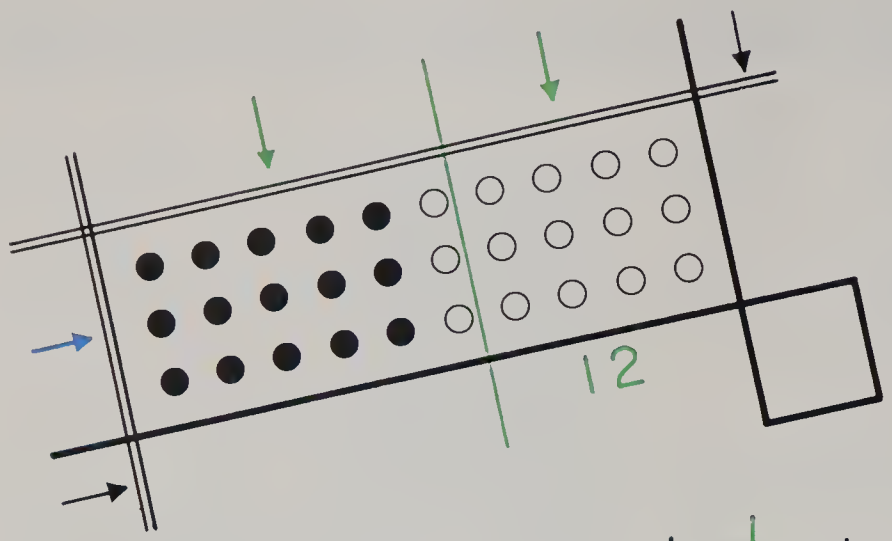
19	24	
20	23	
21	22	

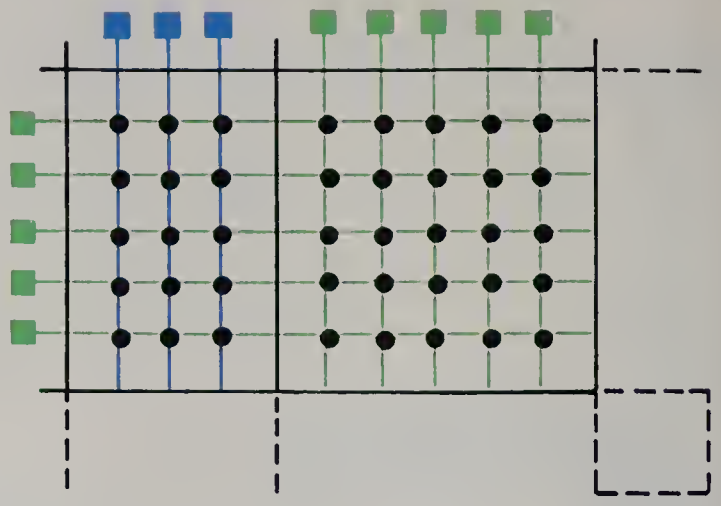
"Ah hah!" exclaimed Numo, "Now I can find the sum of all the whole numbers 1 through 100.

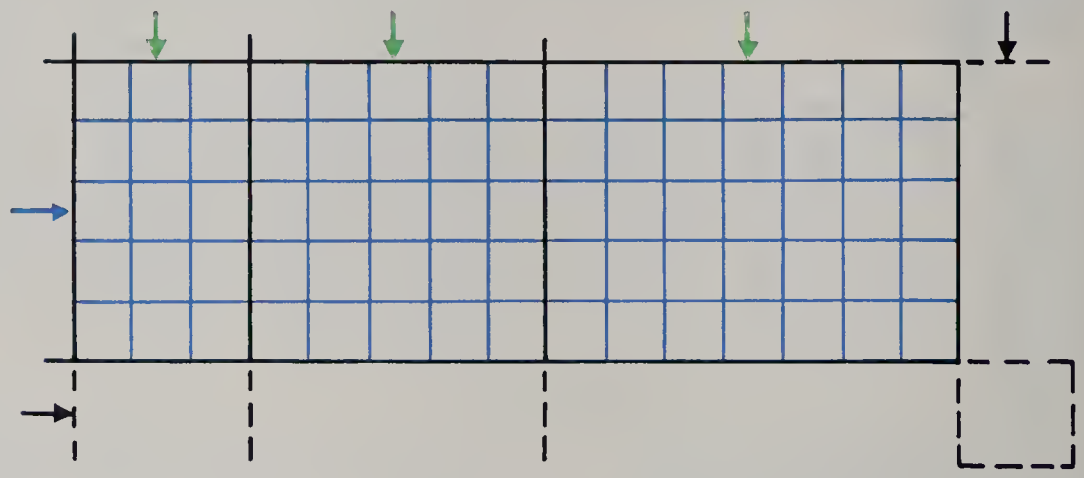
H

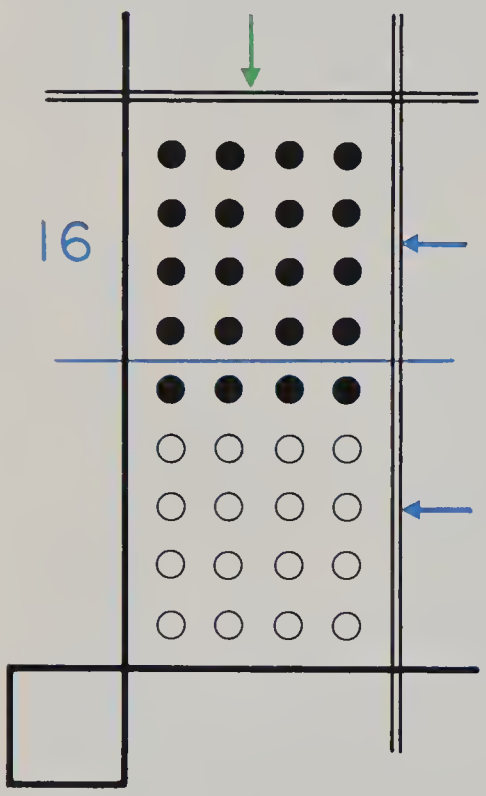
1	2	3	4	5	6	7	45	46	47	48	49	50
100	99										52	51

Here are more sketches from Numo's workbook—

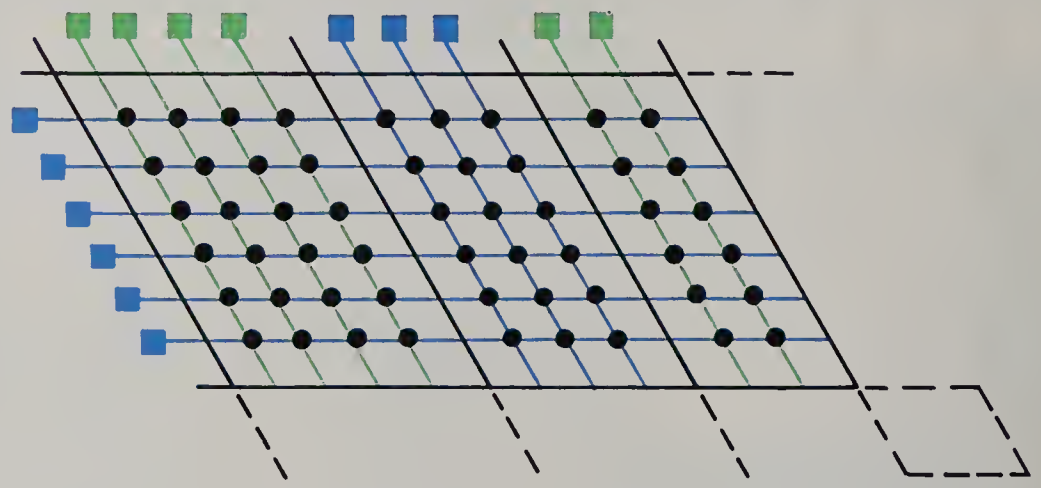




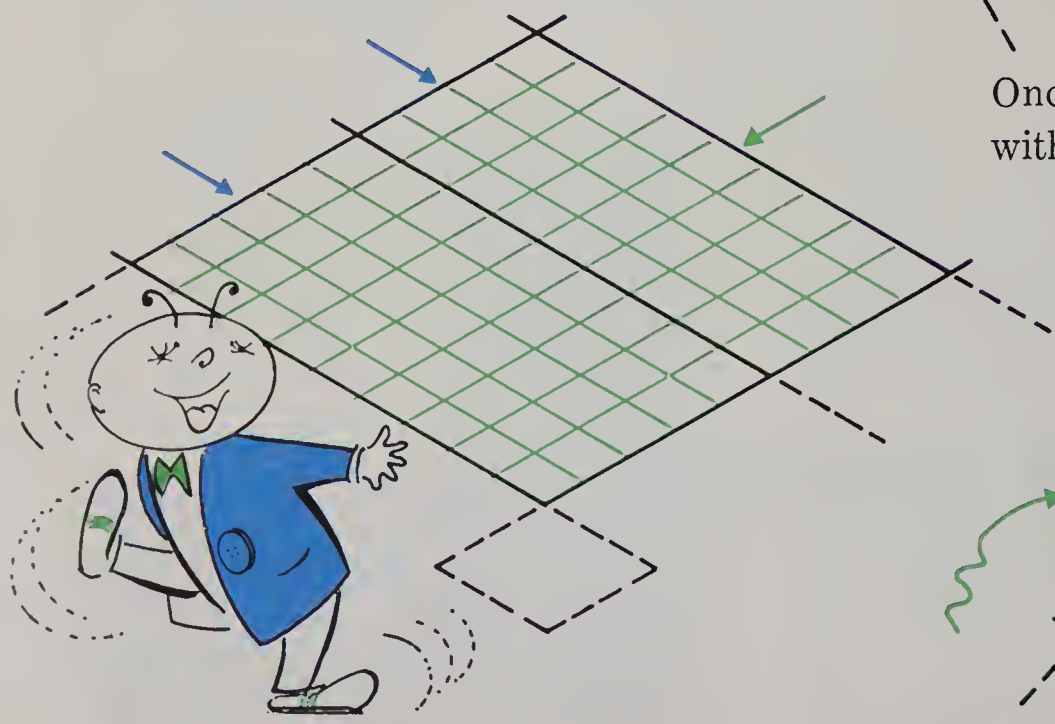


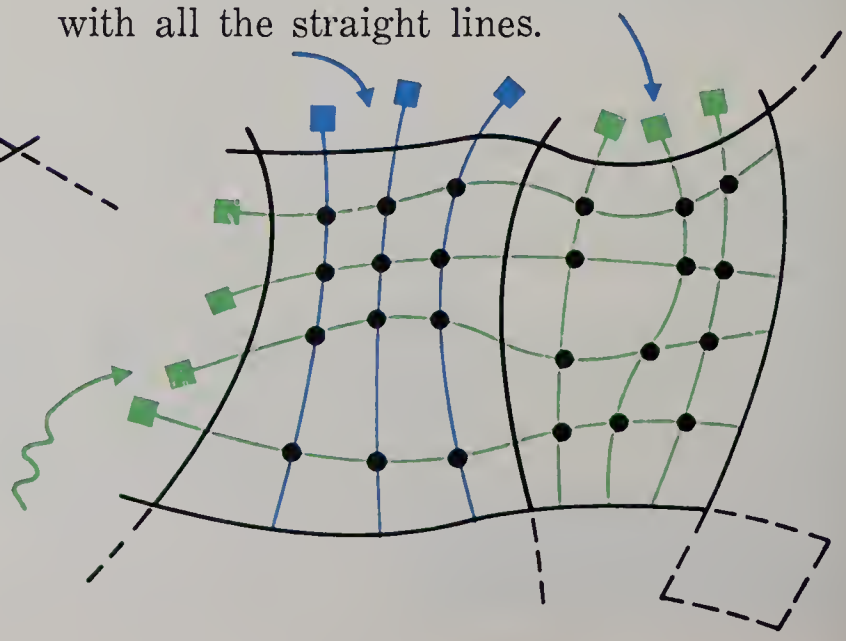


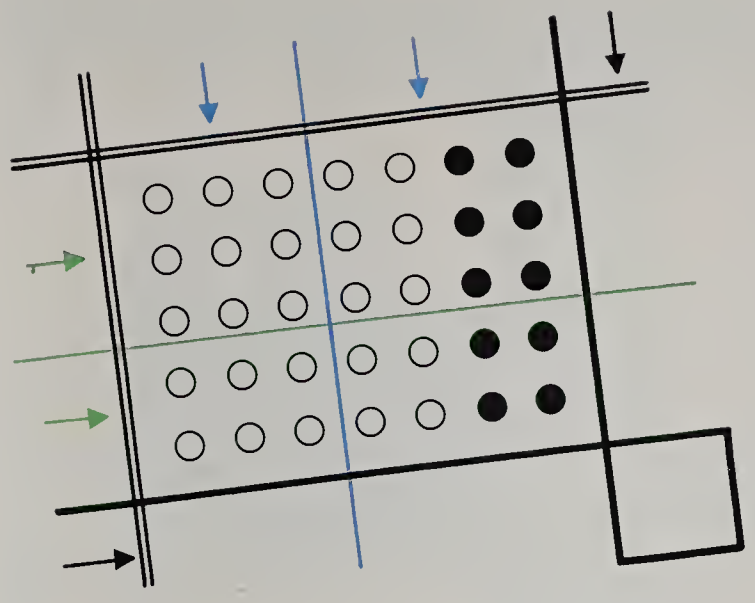
Numo liked to twist his puzzles.

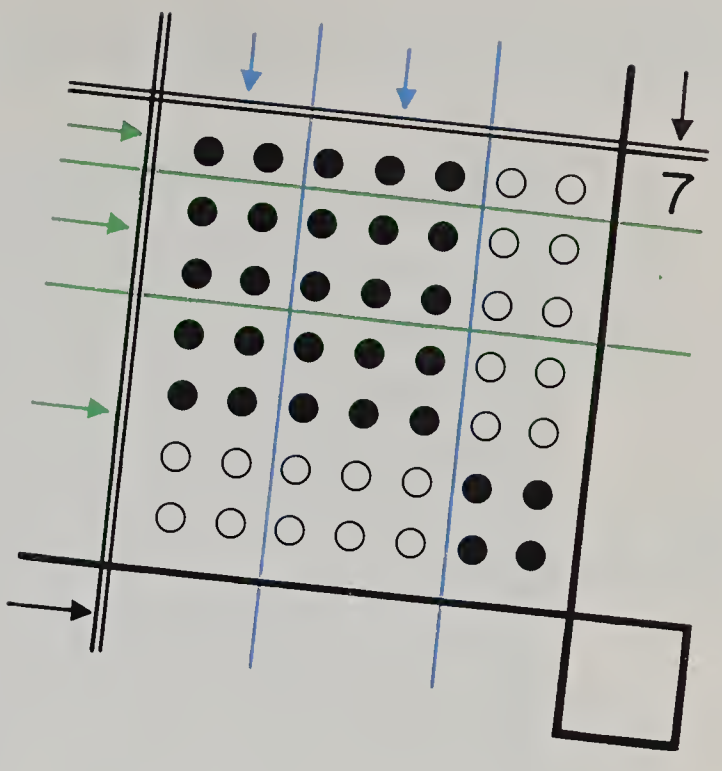


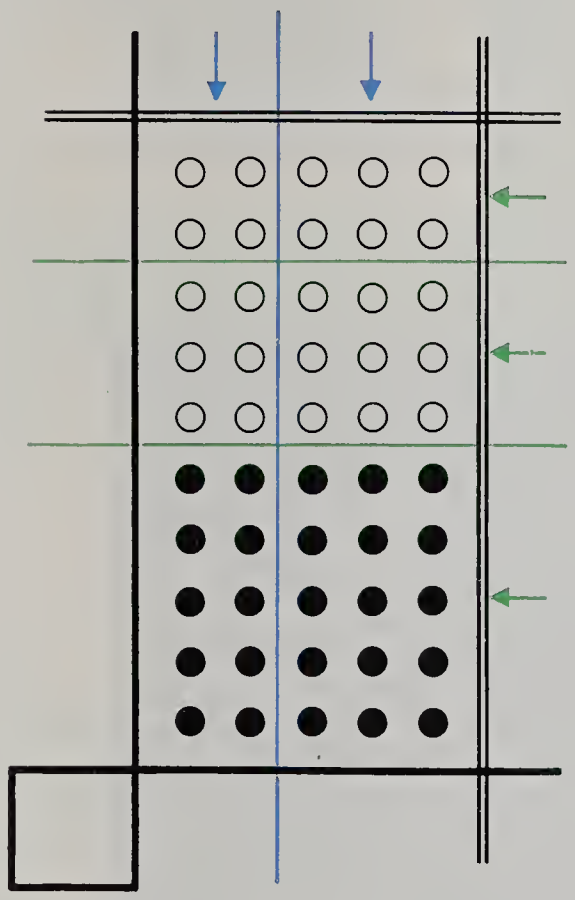
Once Numo became very tired with all the straight lines.

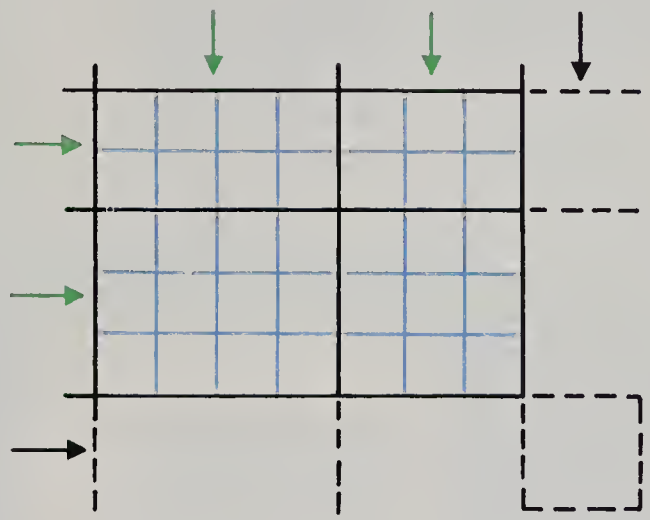


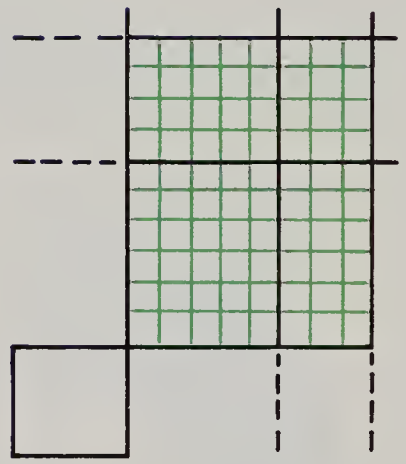




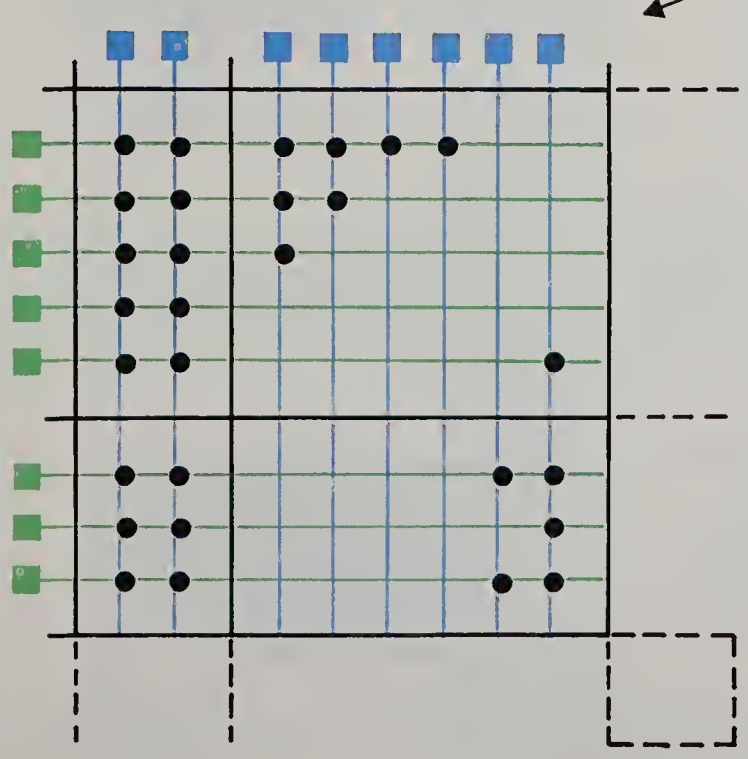


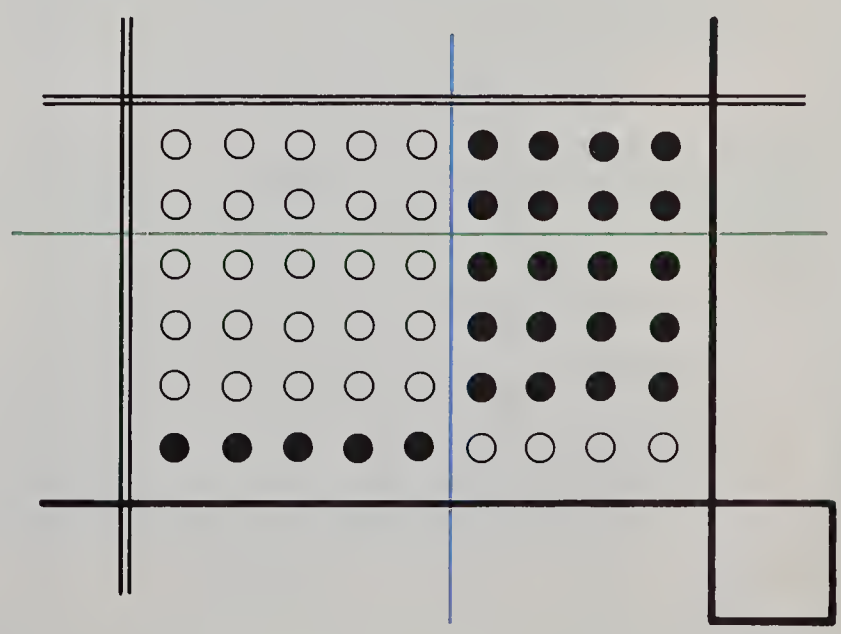




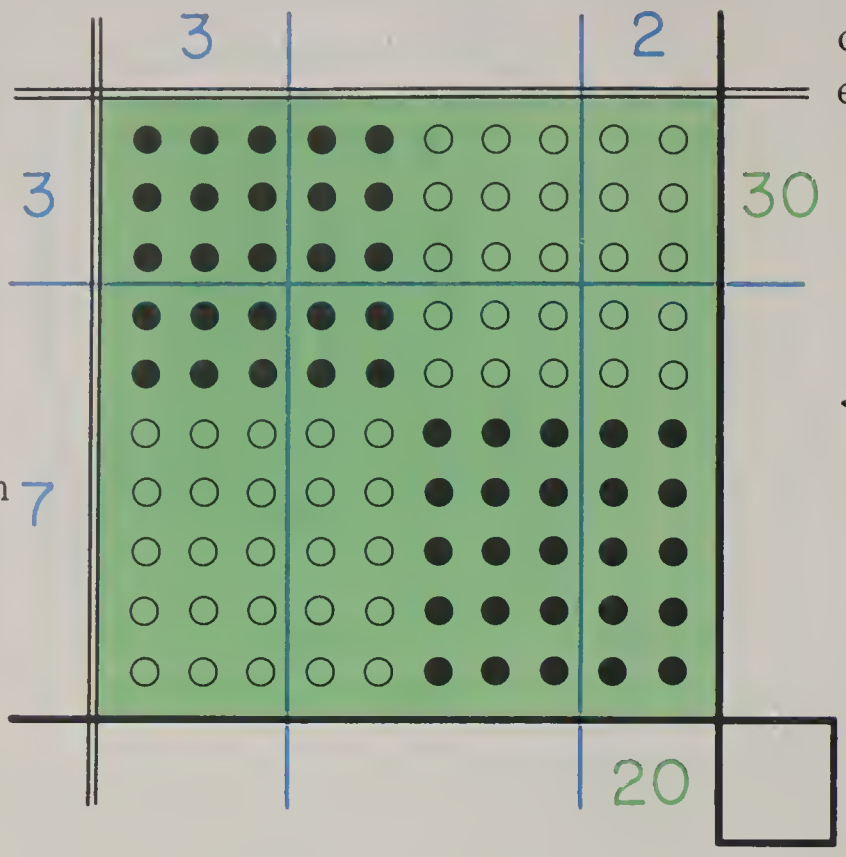


In this corner Numo became tired of putting in the black dots to count. He hurried on to something else. It must be that he wanted you to finish the dots as well as the numbers.

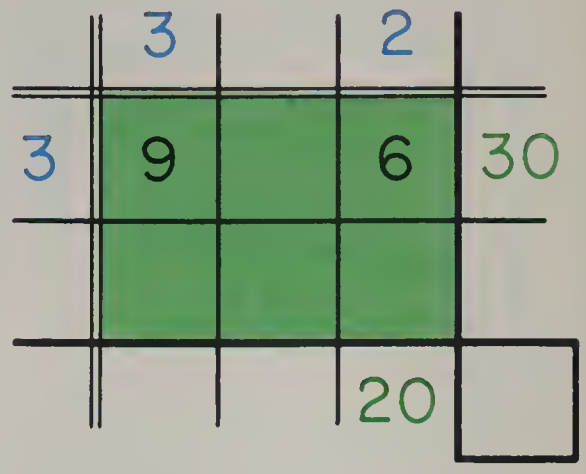


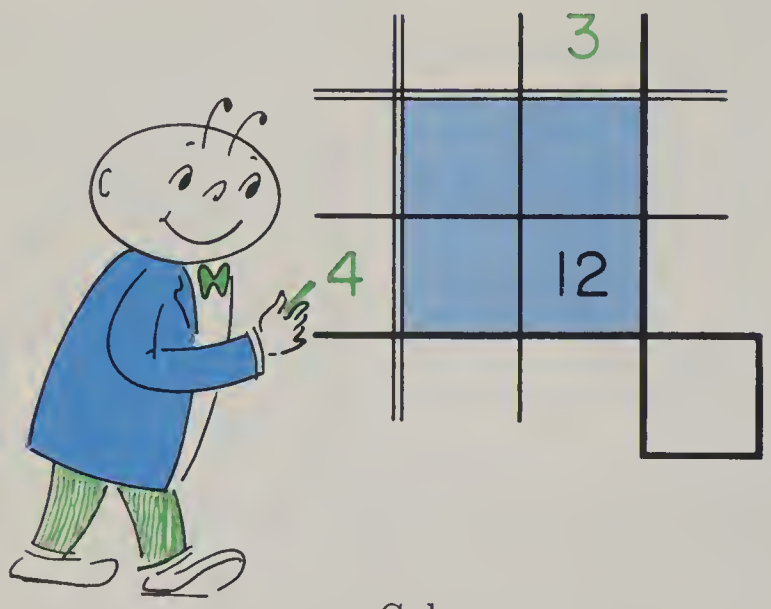


Columns in each section

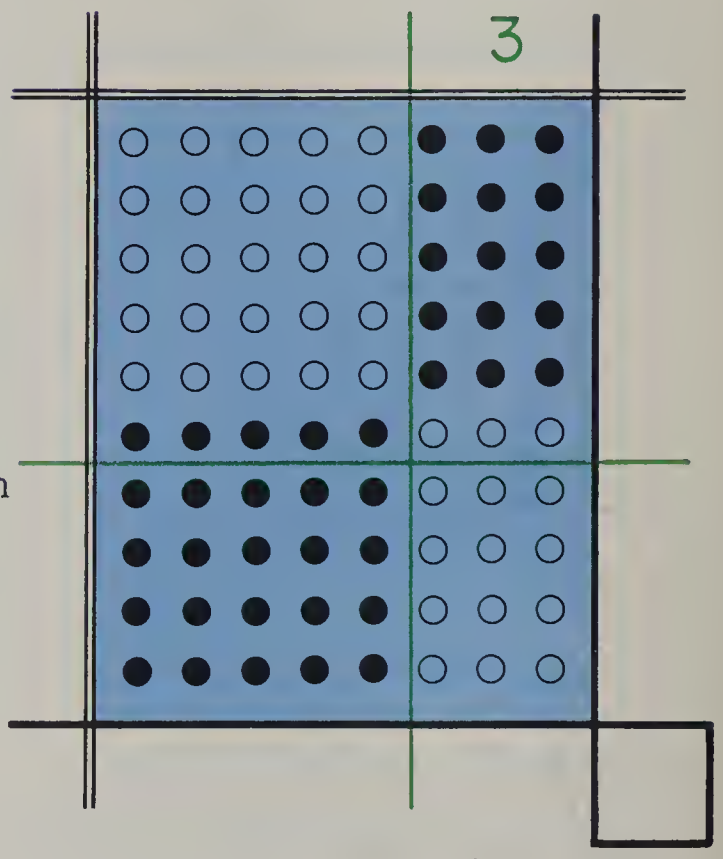


Numo uses the cross number puzzle to help count large number of dots. Then he put ears on it again.

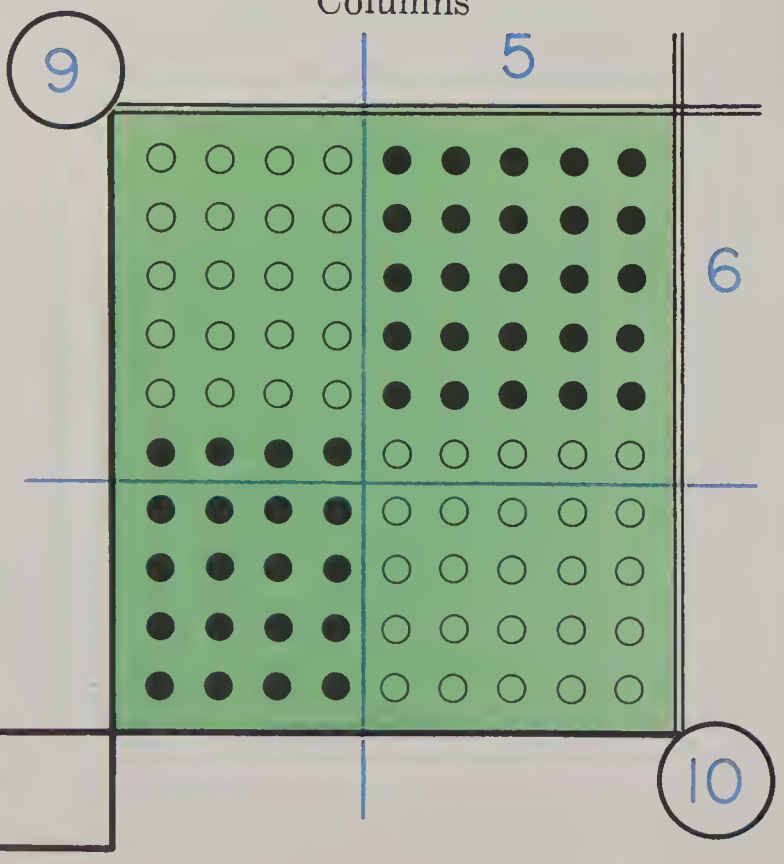


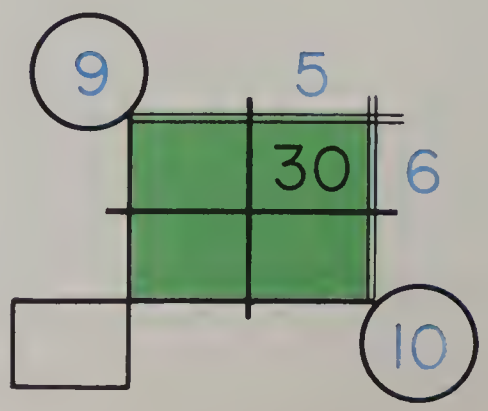


Columns in each section



Columns

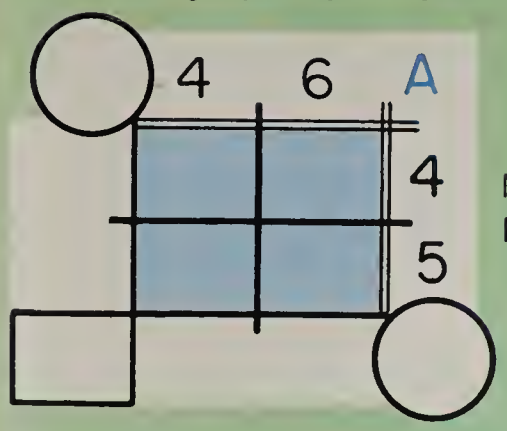


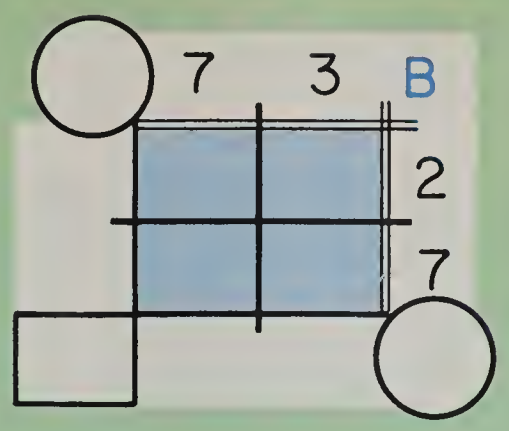


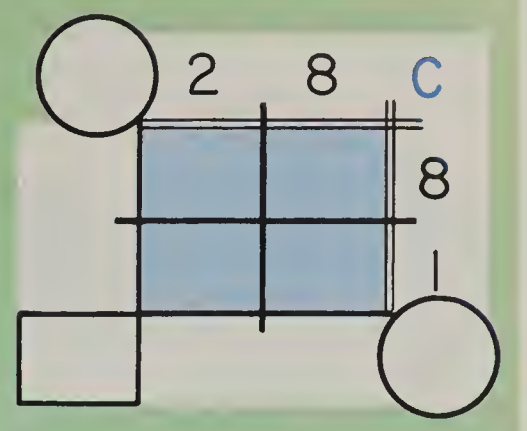
Let's experiment with that last idea from the scrapbook.

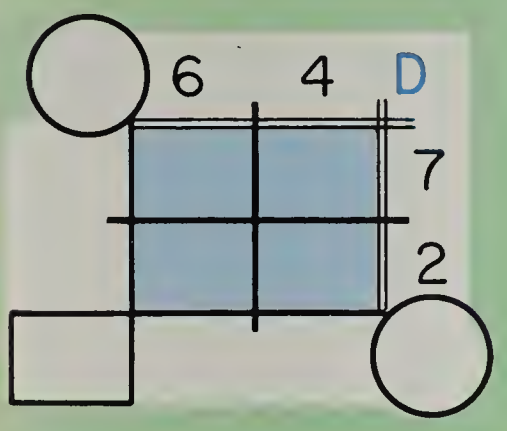
No. of Columns

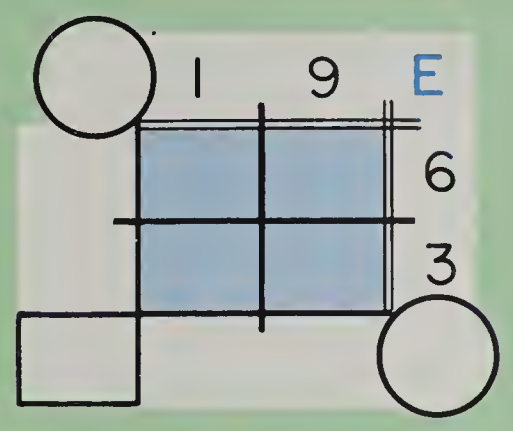
No. of Rows

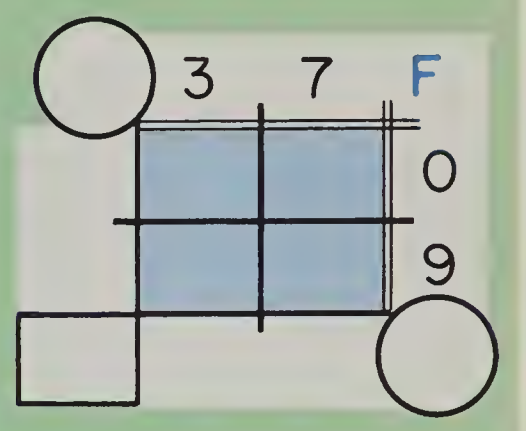


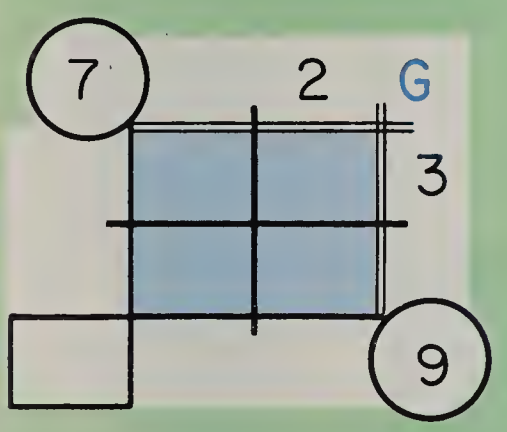


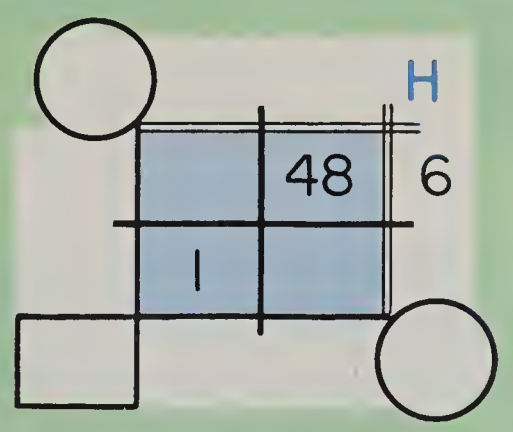


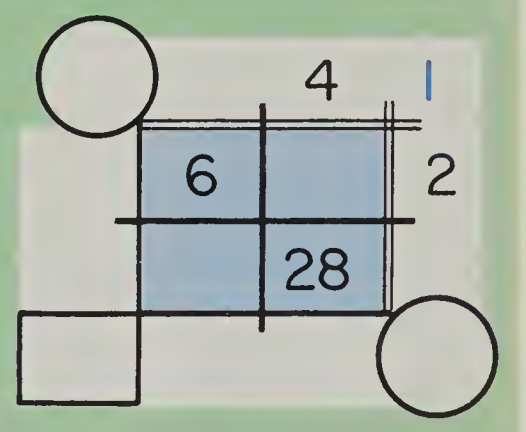


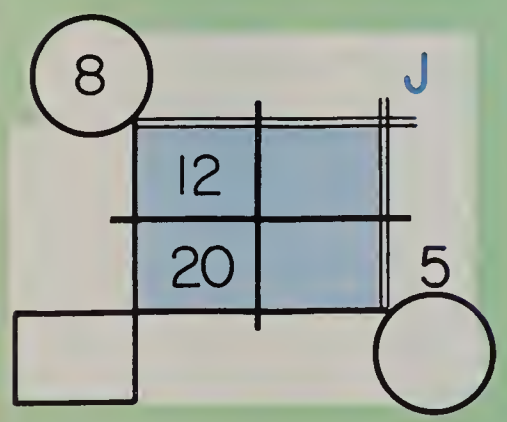


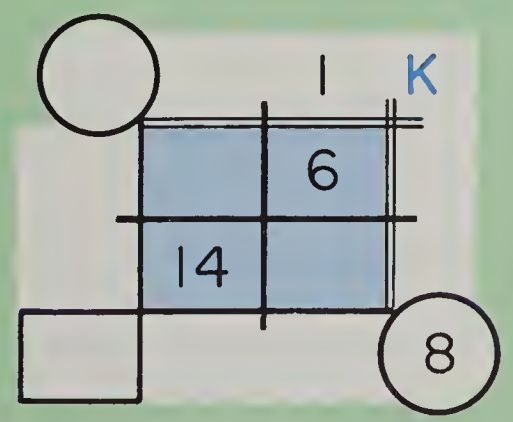


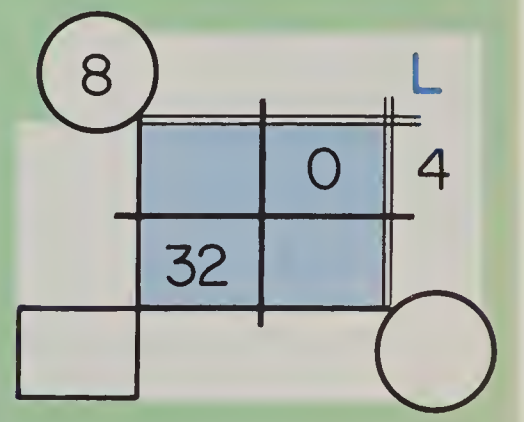


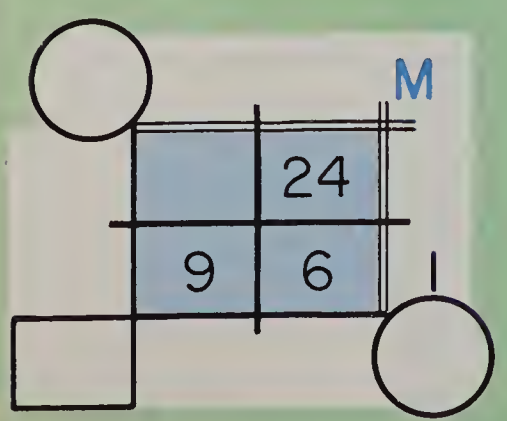


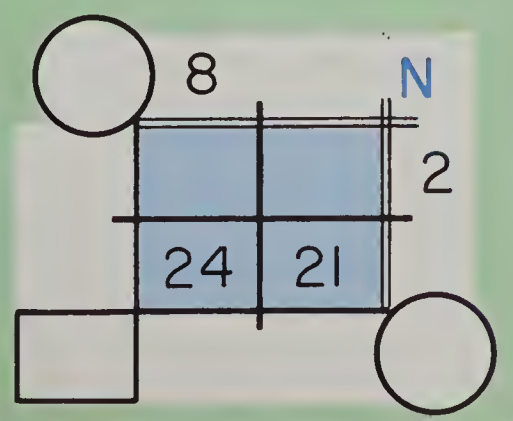


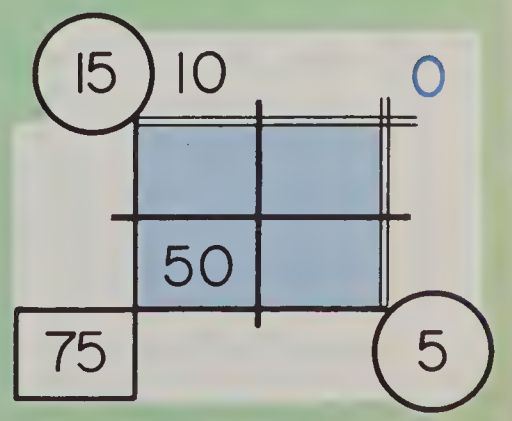




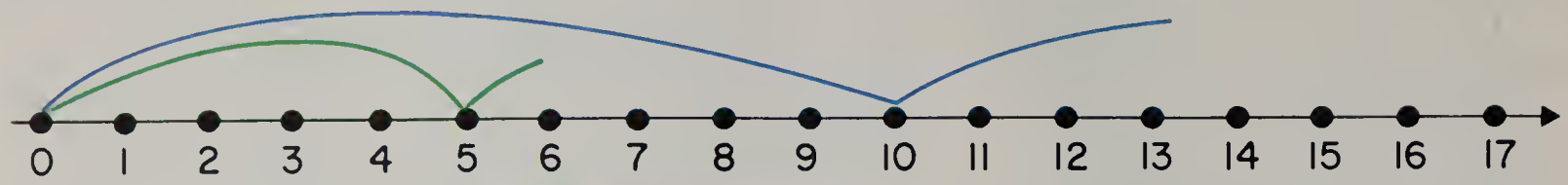


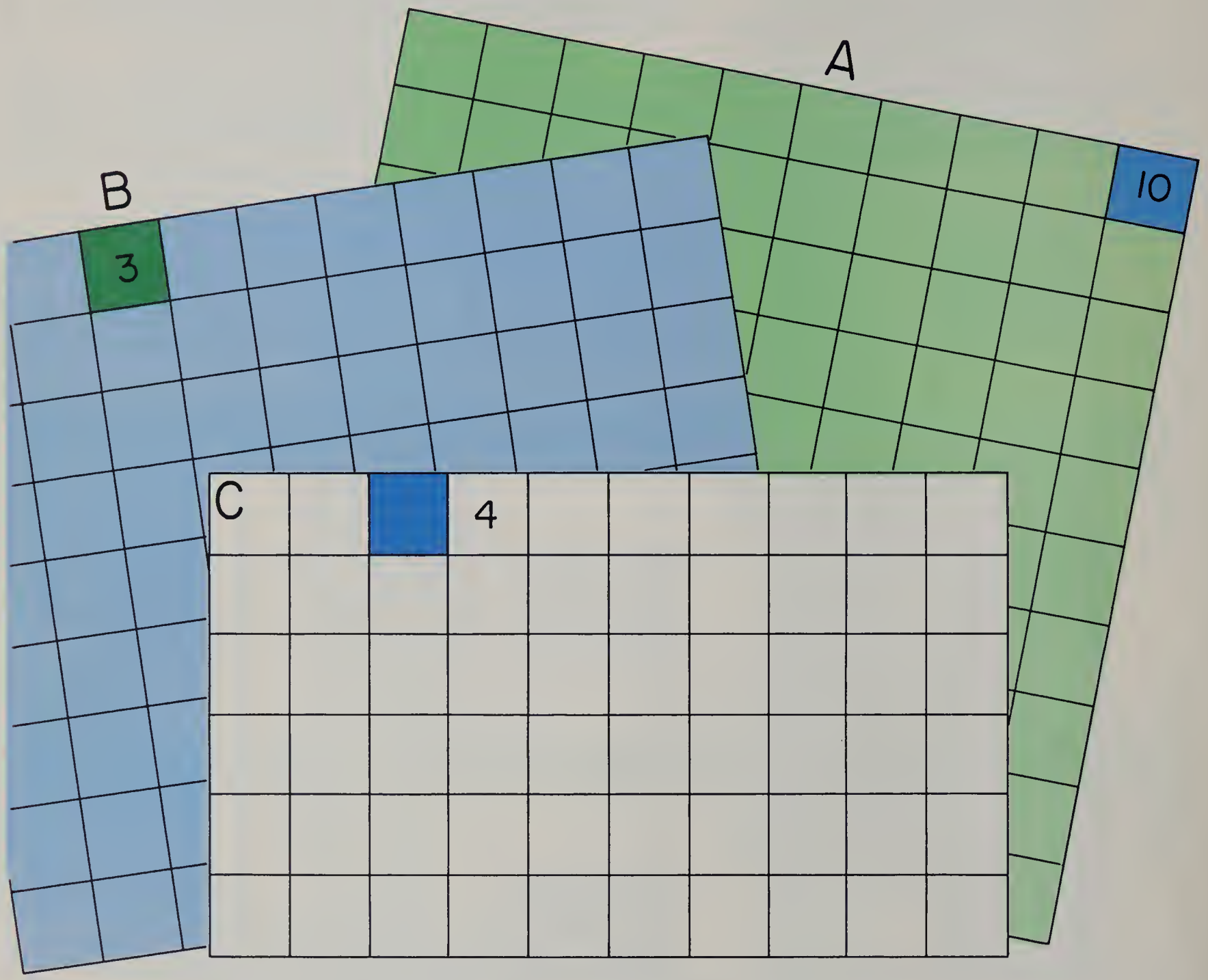




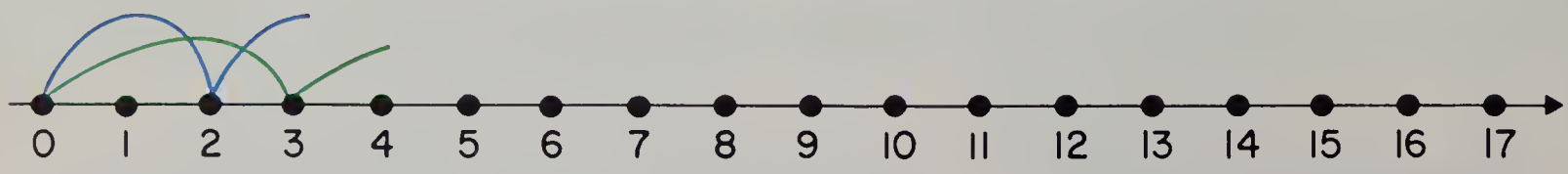


Start at 0 . Take jumps of 5. In chart A write names of landing points.
Start at 0 . Take jumps of 10. In chart A color landing points blue.

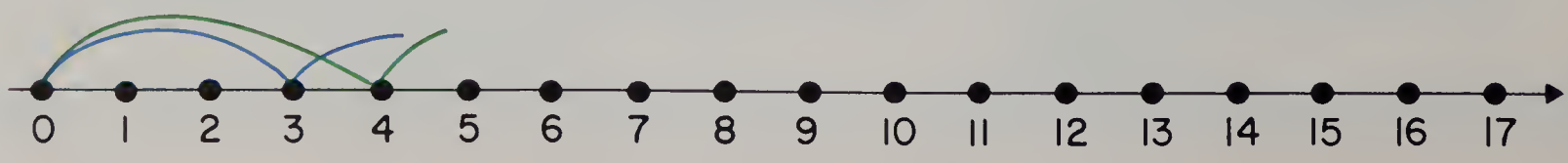




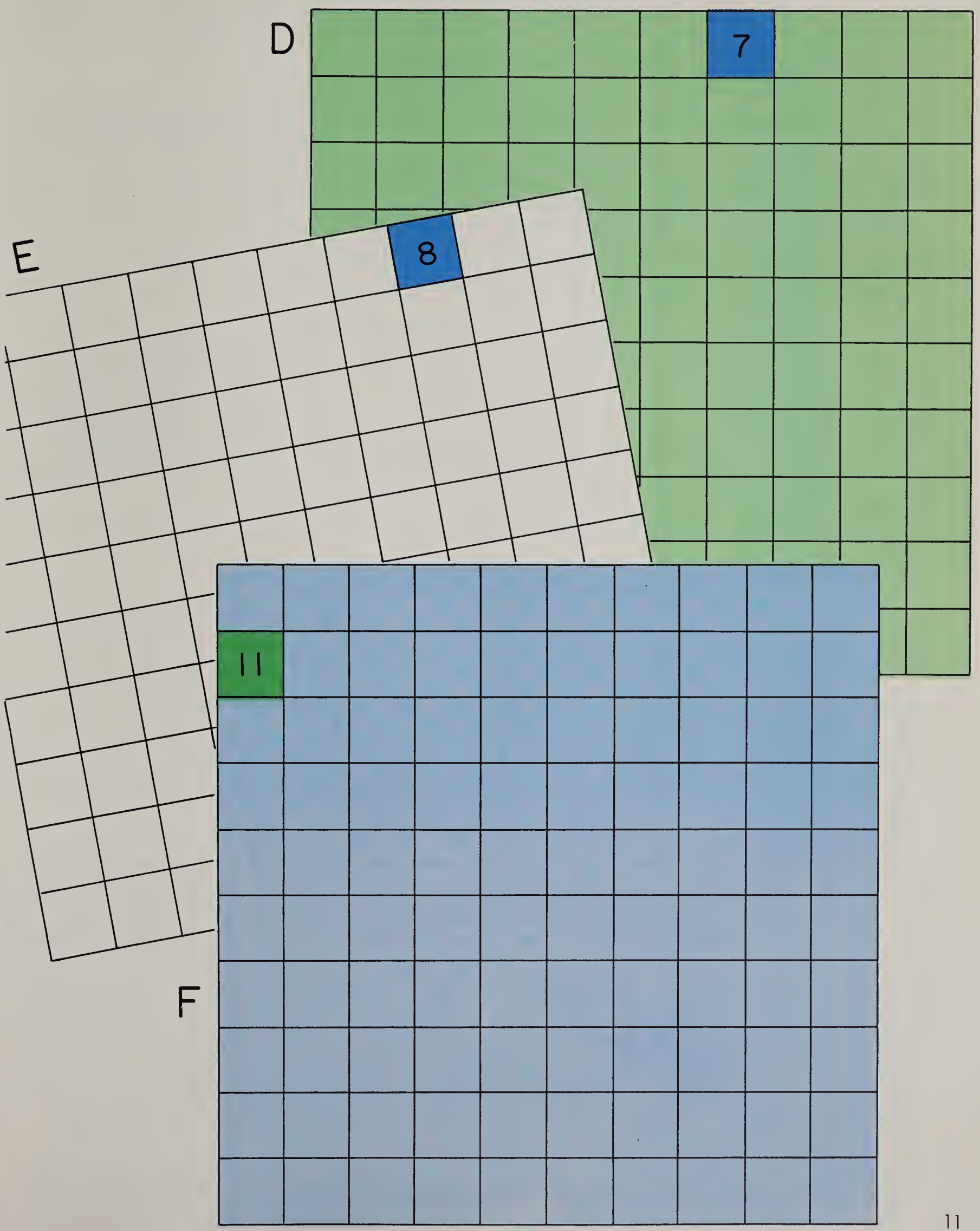
Start at 0 . Take jumps of 2. In chart B write names of landing points.
Start at 0 . Take jumps of 3. In chart B color landing points green.



Start at 0 . Take jumps of 4. In chart C write names of landing points.
Start at 0 . Take jumps of 3. In chart C color landing points red.



On D, record jumps of 2 by writing names of landing points. Color jumps of 7.
On E, record jumps of 6 by writing names of landing points. Color jumps of 8.
On F, record jumps of 9 by writing names of landing points. Color jumps of 11.



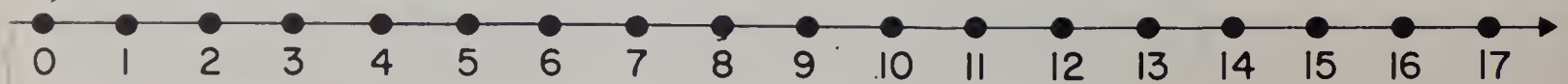
Record results from pages 10 and 11.

Size of each jump

Starting point

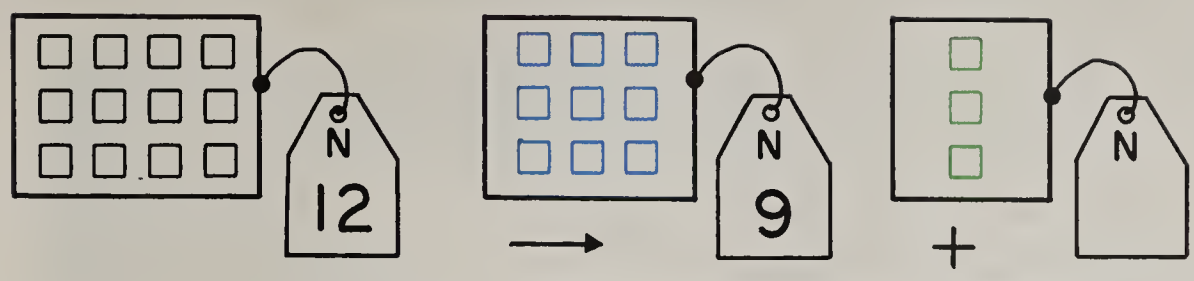
Size of each jump	1st	2nd	3rd	4th	5th	6th	7th	8th	9th
10	0	10	20						
9	0								
8	0								
7	0								
6	0								
5	0	5							
4									
3									
2	0	2	4						
1									

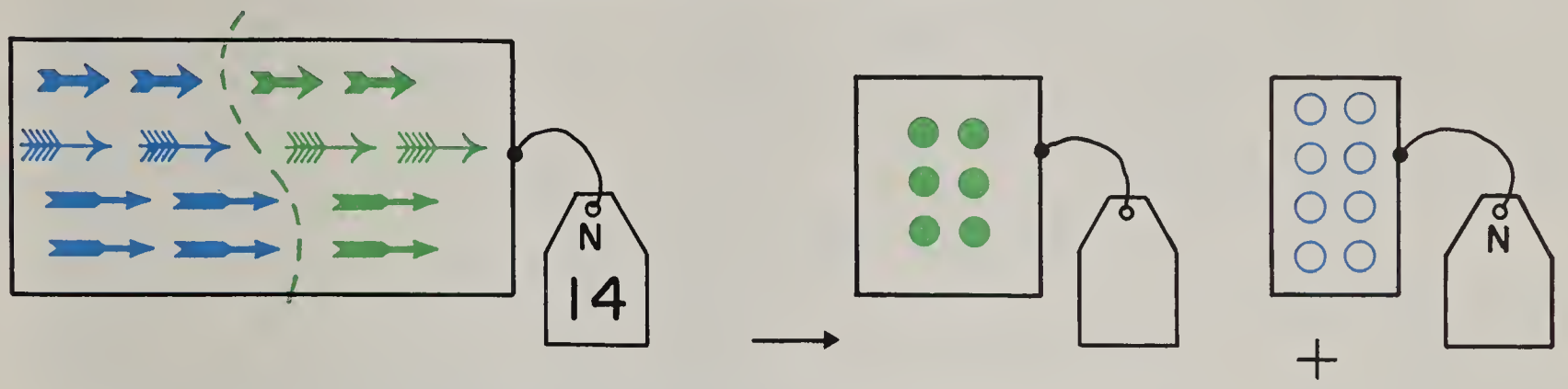
Number of jumps

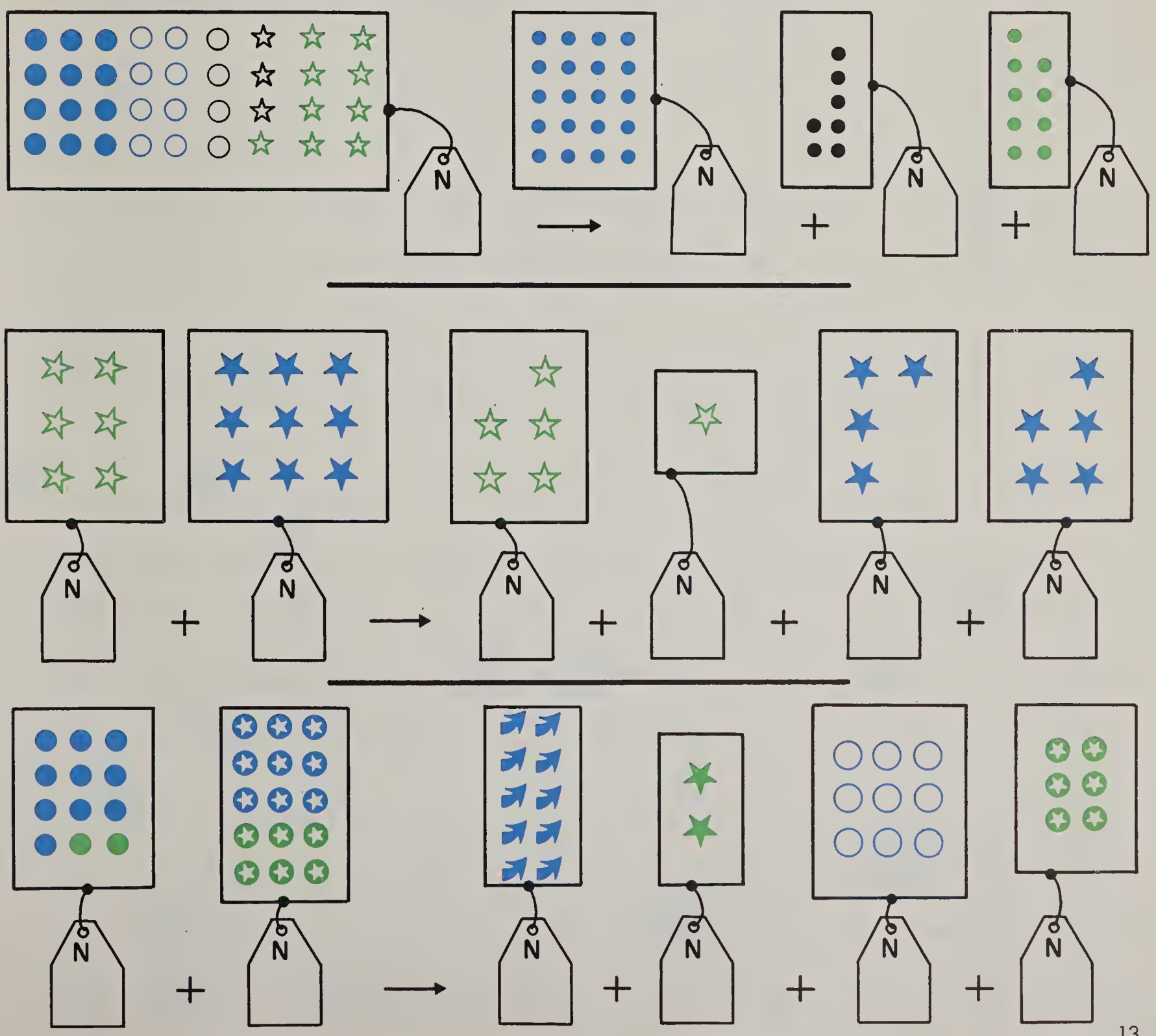


Five Pictures of "REGROUPING"

12→9 + 3 . . . . twelve goes to nine plus 3. Or, 12 objects can be regrouped into a group of 9 and a group of 3.







Let's Play ADDITION and SUBTRACTION GAMES!

Every game begins with rules agreed to by all players.

The game of REGROUPING: Rule A. When a number is circled, we agree to rewrite it as the sum or difference of two numbers.

Examples:

a. (7) → 2 + 5 (Read: 7 goes to 2 + 5)

b. (12) → 8 + __ c. (9) → __ − 3 d. (10) → 3 + __ + 2 e. (17) → __ + __

f. (25) → __ − __ g. (18) → __ + __ + __ h. (25) → __ + __ − __ i. (50) → __ + __

The game becomes more fun.

j. (7) + (8) → __ − __ + __ + __ k. (12) + (13) → __ + __ + __ + __ − __

l. (4) + (18) → __ − __ + __ − __ − __ m. (14) + (29) → __ + __ + __ + __ + __

Be sure to regroup all numbers circled.

n. (12) + (19) → __ + __ + __ − __ o. (18) + (18) → __ − __ + __ + __

p. 21 + (12) → __ + __ − __ q. (31) + (9) → __ + __ + __ + __ + __

r. (15) + (15) → __ + __ + __ − __ s. (19) + (31) → __ − __ + __ + __

t. (32) + (23) → __ − __ + __ − __ u. (27) + (43) → __ − __ + __ + __

Rule B: Where a place is marked by a blue line or a blue box, we agree that a multiple of 10 must be written there. Where there is a green line or a green box, we agree that we must write a number less than 10.

Give examples of numbers that are multiples of 10: __ , __ , ■ , __ , ■

Give examples of numbers that are less than 10: __ , ■ , __ , __ , __

A. (17) → ■ + __ B. (9) → __ − __ C. (21) → __ + __ + __

D. (15) → __ − ■ E. (12) → ■ + ■ F. (16) → __ − __ + __

G. (10) → __ − __ H. (50) → __ + __ I. (20) → ■ + __ − __

J. (10) → __ + __ − __ K. (75) → __ + __ L. (50) → __ + ■ + __

M. (15) + (12) → ■ + __ + __ + __ N. (17) + (13) → __ − __ + __ + __

O. (23) + (18) → __ + __ + __ + ■ P. (24) + (7) → __ + __ + ■ + __ + __

Q. (36) + (29) → __ + __ + __ + __ R. (29) + (49) → ■ + __ + ■ − __

Rule C: In grouping or regrouping, use only the facts of Arithmetic included in Table I.

Table I

+	0	1	2	3	4	5	6	7	8	9	10
10	10	11	12	13	14	15	16	17	18	19	20
9	9	10									19
8	8	9	10								18
7	7	8	9	10							17
6	6	7	8	9	10						16
5	5	6	7	8	9	10					15
4	4	5	6	7	8	9	10				14
3	3	4	5	6	7	8	9	10			13
2	2	3	4	5	6	7	8	9	10		12
1	1	2	3	4	5	6	7	8	9	10	11
0	0	1	2	3	4	5	6	7	8	9	10

		Does Table I show result? Yes or No
1.	3 + 5 →	Yes
2.	10 + 9 →	
3.	8 + 7 →	No
4.	(13) → ___ + ___	
5.	8 − 3 →	Yes
6.	12 − 3 →	
7.	17 − 10 →	

In example 3, we can use Table I several times to reach a result.

8. 8 + 7 → 5 + 3 + 7 → 5 + 10 → ___ ; or we could say

9. 8 + 7 → 8 + ■ + ■ → ■ + 5 → ___

10. 5 + 6 → 5 + ___ + ___ → ___ + ___ → ___

11. 5 + 6 → 1 + ___ + ___ → ___ + ___ → ___

12. 4 + 9 → 3 + ___ + ___ → ___ + ___ → ___

13. 4 + 15 → ___ + ___ + 10 → ___ + ___ → ___

14. 14 − 5 → ___ + 10 − ___ → 4 + ___ → ___

15. 11 − 9 → ___ + ___ − ___ → ___ + ___ → ___

Do the rest on your own. Remember, use only the facts listed in Table I.

16. 5 + 7 → _____

17. 9 + 8 → _____

18. 2 + 9 → _____

19. 13 − 5 → _____

20. 4 + 11 → _____

Henry says: "When you are adding numbers, it does not matter in which order you do it." So, Henry suggested Rules D and E. Do you think these are sensible?

Rule D: The order of any pair of terms that is separated by + may be reversed.

Example: a. $5 + 3 \rightarrow 3 + 5$ b. $5 + 3 + 1 \rightarrow 3 + 5 + 1$

Rule E: The above Rule I may be used repeatedly.

Example: c. $5 + 3 + 1 \rightarrow 3 + 5 + 1 \rightarrow 3 + 1 + 5 \rightarrow 1 + 3 + 5$

Using Rules I and II, change the order to that shown in blue

Change expressions to this order

d. 1 + 2 + 3 → 2 + __ + __ → __ + __ + __ → __ + __ + __ (3+2+1)

e. 1 + 2 + 3 → __ + 3 + __ → __ + __ + __ → __ + __ + __ (3+2+1)

f. 2 + 4 + 6 → 4 + __ + __ → __ + __ + __ → __ + __ + __ (6+4+2)

g. 2 + 4 + 6 → __ + __ + __ → __ + __ + __ → __ + __ + __ (6+4+2)

h. 1 + 5 + 4 → __ + __ + __ → __ + __ + __ → __ + __ + __ (4+5+1)

i. 3 + 7 + 10 → + + → + + → + + (10+7+3)

Try writing these long sentences in the form some poetry is written down:

j. 1 + 2 + 3 →
__ + __ + __ →
__ + __ + __ →
3 + 2 + 1

k. 8 + 10 + 9 →
__ + __ + __ →
9 + 8 + 10

l. 7 + 1 + 3 →
__ + __ + __ →
1 + 3 + 7

m. 1 + 2 + 3 + 4 →
2 + __ + 3 + __ →
__ + __ + __ + __ →
2 + 3 + 4 + 1

n. 1 + 3 + 4 + 7 →
__ + __ + __ + __ →
3 + 1 + 7 + 4

o. 10 + 15 + 5 + 20 →
__ + __ + __ + __ →
5 + 10 + 15 + 20

p. 1 + 2 + 3 + 4 →
__ + __ + __ + __ →
__ + __ + __ + __ →
__ + __ + __ + __ →
__ + __ + __ + __ →
__ + __ + __ + __ →
4 + 3 + 2 + 1

q. Can you find examples on this page in which → can not be replaced by = ?

Using rules D and E, can you rearrange the following expressions into any order you want?

a. 3+7 →
b. 1+10+9 →
c. 2+4+6+8 →
d. 1+3+5+7+9+5 →
e. 1+2+24+23 →

Of what practical use would it be to rearrange b, c, d and e as follows?

b. 1+9+10 →
c. 2+8+4+6 →
d. 1+9+3+7+5+5 →
e. 1+24+2+23 →

Rearrange the following any way you like in your mind and write down the final results directly.

f. 2+10+8 →
g. 2+6+4+8 →
h. 5+9+7+3+1+5 →
i. 5+7+5 →
j. 8+9+10+2+1 →
k. 3+9+7 →
l. 5+2+5+8+6 →
m. 19+9+1+1 →
n. 15+4+5+16 →
o. 18+7+2+10 →
p. 23+9+7+11 →
q. 13+19+12+1+8+7 →
r. 5+10+15+20+25 →
s. 17+6+19+8 →
t. 5+6+7+8+9+10 →
u. 3+5+7+9+11+13+15+17 →
v. 11+10+9+8+7+6+5+4 →

John asks, "What would happen if we used these rules when we had some subtractions?"

$$5 - 3 \rightarrow 3 - 5.$$

This would lead us to use a → where we could not replace → by = , because $5 - 3 = 3 - 5$ is a false statement.

Mark the following examples of change in order as true or false if → is replaced by = .

		T	F
1.	7 - 2 + 4 → 7 - 4 + 2		
2.	7 - 2 + 4 → 7 + 4 - 2		
3.	7 - 2 + 4 → 4 + 2 - 7		
4.	8 - 6 - 1 → 8 - 1 - 6		
5.	8 + 6 - 1 → 8 - 1 + 6		
6.	10 - 5 + 7 →	✓	
7.	10 - 5 + 7 →		✓
8.	10 - 5 + 7 →	✓	
9.	5 + 6 - 3 →	✓	
10.	5 + 6 - 3 →	✓	
11.	5 + 6 - 3		✓
12.	4 - 2 - 1	✓	

Using the regrouping rule A on page 14, mark the following true or false if → is replaced by = .

		T	F
13.	5 - ③ → 5 - 2 + 1		
14.	5 - ③ → 5 - 2 - 1		
15.	10 - ⑦ → 10 - 5 + 2		
16.	10 - ⑦ → 10 - 4 - 3		
17.	10 - ⑦ → 10 - 1 - 6		
18.	⑥ - 4 → 2 + 4 - 4		
19.	8 - ⑦ → 8 - 4 + 3		
20.	8 - ⑤ → 8 - 4 + 1		
21.	8 - ⑦ → 8 - 4 - 3		

On the next two pages use the following rules:

a. ___ only multiples of 10 (including 0) can be written here.

b. ___ only numbers less than 10 (including 0) can be written here.

c. ___ only numbers that are not multiples of 10 and not less than 10 can be written here.

Complete the following examples:

1. ___ + 3 = ___
2. ___ + ___ = 27
3. 10 + ___ = ___
4. 35 − ___ = ___

Rearrange and regroup the following according to the rules on the last four pages so the result is in the form of ___ + ___:

1. 17 + 3 → 10 + 7 + 3 → 10 + 10 → 20 + 0
2. 24 + 9 → ___ + ___ + ___ → ___ + ___ → ___ + ___ + ___ → ___ + ___
3. 18 + 16 → ___ + ___ + ___ + ___ → ___ + ___ → ___ + ___ + ___ → ___ + ___
4. 24 + 25 → ___ + ___ + ___ + ___ → ___ + ___ + ___ + ___ → ___ + ___
5. 26 − 7 → ___ + ___ − ___ → ___ + ___
6. 31 − 14 → ___ + ___ − ___ + ___ → ___ − ___ + ___ − ___ → ___ + ___
7. 59 − 26 → ___ + ___ − ___ + ___ → ___ − ___ + ___ − ___ → ___ + ___
8. 37 + 37 → ___ + ___ + ___ + ___ → ___ + ___ → ___ + ___ + ___ → ___ + ___
9. 8 + 54 → ___ + ___ + ___ → ___ + ___ → ___ + ___ + ___ → ___ + ___
10. 93 − 39 → ___ + ___ − ___ + ___ → ___ − ___ + ___ − ___ → ___ + ___

Below are similar examples written in a form used sometimes in poetry:

a. 93 − 39 →
80 + 13
−30 − 9 →
50 + 4

b. 37 + 54 →
___ + ___
+ ___ + ___ →
___ + ___ →
___ + ___ + ___ →
___ + ___

c. 74 + 18 →
___ + ___
+ ___ + ___ →
___ + ___ →
___ + ___ + ___ →
___ + ___

d. 58 − 29 →
___ + ___
− ___ − ___ →
___ + ___

e. 87 + 7 →
___ + ___
___ + ___ →
___ + ___ →
___ + ___ + ___ →
___ + ___

f. 61 − 8 →
___ + ___
− ___ →
___ + ___

g. 90 − 67 →
___ + ___
− ___ − ___ →
___ + ___

h. 99 + 99 →
___ + ___
+ ___ + ___ →
___ + ___ →
___ + ___ + ___ →
___ + ___

A Subtraction Game: Rule — Suppose that all we know about subtraction is that any number subtracted from itself equals 0 . . . $7-7=0$, $13-13=0$, etc.

1. $5+3-3 \rightarrow 5+0 \rightarrow 5$

2. $9-7 \rightarrow 2+\_-7 \rightarrow 2$

3. $3+5-3 \rightarrow 5+\_-\_ \rightarrow \_$

4. $9-3+7 \rightarrow 6+\_-\_+\_ \rightarrow \_+\_ \rightarrow \_$

5. $8-3-4 \rightarrow \_+\_-\_-\_ \rightarrow \_-\_ \rightarrow \_+\_-\_ \rightarrow \_$

6. $9-1-6 \rightarrow \_+\_-\_-\_ \rightarrow \_-\_ \rightarrow \_+\_-\_ \rightarrow \_$

7. $15-7-8 \rightarrow \_+\_-\_-\_ \rightarrow \_-\_ \rightarrow \_$

8. $13-4-5 \rightarrow \_+\_-\_-\_ \rightarrow \_-\_ \rightarrow \_+\_-\_ \rightarrow \_$

The following examples require some rearranging:

9. $3-5+6 \rightarrow \_+\_-\_ \rightarrow \_+\_+\_-\_ \rightarrow \_+\_ \rightarrow \_$

10. $2-4+3 \rightarrow \_+\_-\_ \rightarrow \_-\_ \rightarrow \_+\_-\_ \rightarrow \_$

11. $0-5+6 \rightarrow \_+\_-\_ \rightarrow \_-\_ \rightarrow \_+\_-\_ \rightarrow \_$

12. $7-8+1 \rightarrow \_+\_-\_ \rightarrow \_-\_ \rightarrow \_$

13. $3-5+7 \rightarrow \_+\_-\_ \rightarrow \_+\_+\_-\_ \rightarrow \_+\_ \rightarrow \_$

14. $8-3-1 \rightarrow \ +\ -\ -\ \rightarrow\ -\ \rightarrow\ +\ -\ \rightarrow$

15. $3-8+6 \rightarrow \_+\_-\_ \rightarrow \_-\_ \rightarrow \_+\_-\_ \rightarrow \_$

Mark the following statements as True or False:

		T	F
a	$3+7=1+3+7-1$		
b	$27=4+27-4$		
c	$8-6=8-1+6+1$		
d	$4-1=1+4-1+1$		
e	$7+10=7-6+10+6$		
f	$13-3=8+13-8-3$		
g	$19+38=19+1+38-1$		
h	$38-19=38+1-19+1$		
i	$68+25=2+68+25-2$		
j	$37+37=37-3+37+3$		
k	$34+56=23+4-25+6$		
l	$25+35=25+5+35-5$		
m	$29+13=13+6+29-6$		
n	$87-68=3+87-68+3$		
o	$44-35=44-5-35+5$		

		T	F
A	$24+39=23+40$		
B	$85-7=88-10$		
C	$25+7=28+10$		
D	$71-71=70-70$		
E	$21+21=20+20$		
F	$93-87=90-84$		
G	$93-87=96-90$		
H	$97-68=86-79$		
I	$80-39=81-40$		
J	$145-39=146-40$		
K	$262+98=260+100$		
L	$321-182=339-200$		
M	$268+286=270+280$		
N	$1003-995=1008-1000$		
O	$555+950=605+900$		

Ben's First Arithmetic Class at Roosevelt School

Ben Harris moved to Central City and attended Roosevelt School.

When arithmetic class began, Mrs. Barnes asked, "Have you had much work in multiplying, Ben?"

"About all I know is that zero times 5, 4, 3, 2, or 1 is 0; and that one times any number doesn't change them any. Also if $4 \times 3 = 12$, then $3 \times 4 = 12$, and this works with any pair of factors."

"Do you know anything about division?", Mrs. Barnes asked.

"All I know is that anything I can do by multiplying, I can undo by dividing. I can look at my chart and see that $15 \div 3 = 5$ because $5 \times 3 = 15$."

"How much is $35 \div 5$?", Ben was asked.

"It's not on my chart, so I would have to work it out."

Ben's Chart

X	2	3	4	5
5	10	15	20	25
4	8	12	16	20
3	6	9	12	15
2	4	6	8	10

Mrs. Barnes was puzzled. "Don't you know that $35 \div 7 = 5$ because $5 \times 7 = 35$?"

"I guess I stopped at 5×5. I don't remember how many 5×7 is, but I could work it out."

$5 \times 7 = 5 \times (4 + 3) = 20 + 15 = 35$

"The arrows mean Do-It-To-Both; multiply both 4 and 3 by 5. I could also do it this way."

$5 \times 7 = 5 \times (2 + 5) = 10 + 25 = 35$

"Could you do it this way?", asked Mrs. Barnes.

$5 \times 7 = 5 \times (6 + 1) = 30 + 5 = 35$

"No," Ben admitted, "because 5×6 is not on my chart. I would find the answer this way."

$5 \times 6 = 5 \times (3 + 3) = 15 + 15 = 30$

"Can you divide 35 by 7?", Joe asked. "Yes. This way."

$35 \div 5 = (25 + 10) \div 5 = 5 + 2 = 7$

"You see, I Do-It-To-Both again. Or, you might say I undo-both, since dividing is simply undoing what we do when we multiply."

Jerry Mason had a question.

"How could you do something like 4×13? You say you don't know the answer to 4×6 or 4×7, so you couldn't work out $4 \times (6 + 7)$. Is that right?"

"I can do $4 \times (6 + 7)$ this way."

$4 \times 6 = 4 \times (5 + 1) = 20 + 4 = 24$

$4 \times 7 = 4 \times (5 + 2) = 20 + 8 = 28$

and $24 + 28 = 52$

"But I have a shorter way to find the answer to 14 × 3. I do it this way."

$$4 \times 13 = 4 \times (5 + 5 + 3) = 20 + 20 + 12 = 52$$

"You see! I Do-It-To-All-Three. Sometimes," Ben added, "I have to Do-It-To-All-Four. Here's an example:"

$$5 \times 18 = 5 \times (5 + 5 + 5 + 3) = 25 + 25 + 25 + 15 = 90$$

"How about 14 × 3?", Sue asked. Ben wrote without saying anything.

$$14 \times 3 = (5 + 5 + 4) \times 3 = 15 + 15 + 12 = 42$$

Mrs. Barnes was impressed. So were others in the class. Some were puzzled.

"Will your Do-It-To-Both always work?"

"It always has," Ben answered.

Mrs. Barnes suggested, "OK, we multiply Ben's way for a while."

"Remember," she added, "you must never use any multiplication or division fact in which any of the factors is more than five."

"I think Do-It-To-Both is a silly idea to use in multiplying and dividing," Anne said as she looked at Ben.

"It works," was Ben's only reply.

"Well, Anne, after we've learned more about Ben's way, we will let you show Ben your way of multiplying and dividing."

DO-IT-TO-BOTH

$2 \times 10 = 2 \times (\_\_\_ + \_\_\_) = \_\_\_ + \_\_\_ = \_\_\_\_$

$10 \times 3 = (\_\_\_ + \_\_\_) \times \_\_\_ = \_\_\_ + \_\_\_ = \_\_\_\_$

$2 \times 9 = \_\_\_ \times (\_\_\_ + \_\_\_) = \_\_\_ + \_\_\_ = \_\_\_\_$

$8 \times 3 = (\_\_\_ + \_\_\_) \times \_\_\_ = \_\_\_ + \_\_\_ = \_\_\_\_$

$20 \div 2 = (\_\_\_ + \_\_\_) \div 2 = \_\_\_ + \_\_\_ = \_\_\_\_$

$18 \div 2 = (\_\_\_ + \_\_\_) \div 2 = \_\_\_ + \_\_\_ = \_\_\_\_$

$30 \div 3 = (\_\_\_ + \_\_\_) \div \_\_\_ = \_\_\_ + \_\_\_ = \_\_\_\_$

$24 \div 3 = (\_\_\_ + \_\_\_) \div \_\_\_ = \_\_\_ + \_\_\_ = \_\_\_\_$

DO-IT-TO-ALL-THREE

$2 \times 14 = 2 \times (\_\_\_\_ + \_\_\_\_ + \_\_\_\_) = \_\_\_\_ + \_\_\_\_ + \_\_\_\_ = \_\_\_\_$

$15 \times 3 = (\_\_\_\_ + \_\_\_\_ + \_\_\_\_) \times \_\_\_\_ = \_\_\_\_ + \_\_\_\_ + \_\_\_\_ = \_\_\_\_$

$30 \div 2 = (\_\_\_\_ + \_\_\_\_ + \_\_\_\_) \div \_\_\_\_ = \_\_\_\_ + \_\_\_\_ + \_\_\_\_ = \_\_\_\_$

$60 \div 5 = (\_\_\_\_ + \_\_\_\_ + \_\_\_\_) \div \_\_\_\_ = \_\_\_\_ + \_\_\_\_ + \_\_\_\_ = \_\_\_\_$

$5 \times 13 = \_\_\_\_ \times (\_\_\_\_ + \_\_\_\_ + \_\_\_\_) = \_\_\_\_ + \_\_\_\_ + \_\_\_\_ = \_\_\_\_$

$48 \div 4 =$ _____

$70 \div 5 =$ _____

Ben's Second Day

The next day, Mrs. Barnes asked Anne what she thought of Ben's way of multiplying and dividing. "You mean that Do-It-To-Both stuff? Why does it work only when one of the factors is 5 or less, such as 4×14. "Can I give Ben a simple problem that he can't do with his Do-It-To-Both unless he knows much more than $5 \times 5 = 25$?" "Surely, Anne. Put your problem on the board."

Anne wrote:

$$8 \times 8 = 8 \times (4 + 4)$$

Betty Grant raised her hand. "I think I can do that one Ben's way. May I try?" Mrs. Barnes said, "Of course." Here is Betty's work:

$$8 \times 8 = 8 \times (4 + 4) = (8 \times 4) + (8 \times 4) =$$
$$8 \times 4 = (4 + 4) \times 4 = 16 + 16 = 32$$
$$\text{So, } (8 \times 4) + (8 \times 4) = 32 + 32 = 64$$

"Is that right, Ben?" Betty asked. "Yes. However, I would have done it with a few less steps." "Will you show us, Ben?"
Here is Ben's work:

$$8 \times 8 = (4 + 4) \times (4 + 4) = 16 + 16 + 16 + 16 = 64$$

"I call that the Double-Do-It-To-Both method." "Why did you pick two factors that are the same number?" said Anne. "How about 7×9?" Ben didn't hesitate.

$$7 \times 9 = (5 + 2) \times (5 + 4) = 25 + 20 + 10 + 8 = 63$$

Anne's face fell. "Well, that's a kind of multiplication that's sure full of addition.

"I learned addition before I knew about multiplication, so I use that instead of remembering so many multiplication facts."

"Will it always work, Ben?" Betty asked.

"I've never known it to fail," was Ben's quiet answer.

"Can we try some examples, Mrs. Barnes?" Jerry wanted to know. "Yes, I'll put some on the board."

Anne looked at Ben. "Does he really know what this is all about?" she wondered.

"Remember," Mrs. Barnes said, "You must imagine that you don't know how to multiply if any factor is greater than 5.

6 x 6 = (4 + 2) (5 + 1) = ___ + ___ + ___ + ___ = ___

7 x 8 = (___ + ___) x (___ + ___) = ___ + ___ + ___ + ___ = ___

9 x 6 = (___ + ___) x (___ + ___) = ___ + ___ + ___ + ___ = ___

9 x 9 = (___ + ___) x (___ + ___) = ___ + ___ + ___ + ___ = ___

8 x 9 = (___ + ___) x (___ + ___) = ___ + ___ + ___ + ___ = ___

6 x 8 = (___ + ___) x (___ + ___) = ___ + ___ + ___ + ___ = ___

10 x 7 = (___ + ___) x (___ + ___) = ___ + ___ + ___ + ___ = ___

8 x 10 = (___ + ___) x (___ + ___) = ___ + ___ + ___ + ___ = ___

10 x 10 = (___ + ___) x (___ + ___) = ___ + ___ + ___ + ___ = ___

"And that's the end! Ben's method doesn't work after $10 \times 10 = 100$. I don't think much of a method that can't handle 9×14."
Ben didn't say a word, but went to the board. Here is his work:

9 x 14 = (5 + 4) x (5 + 5 + 4) = 25 + 25 + 20 + 20 + 20 + 16 = 126

Anne's eyes almost popped out. Here are some more examples:

8 x 15 = (___ + ___) x (___ + ___ + ___) = ___ + ___ + ___ + ___ + ___ + ___ = ___

7 x 12 = (___ + ___) x (___ + ___ + ___) = ___ + ___ + ___ + ___ + ___ + ___ = ___

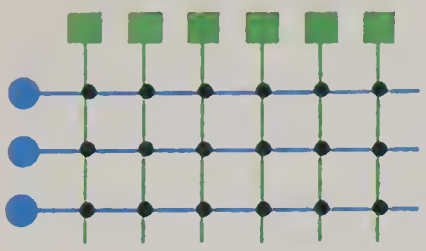
9 x 13 = (___ + ___) x (___ + ___ + ___) = ___ + ___ + ___ + ___ + ___ + ___ = ___

10 x 15 = (___ + ___) x (___ + ___ + ___) = ___ + ___ + ___ + ___ + ___ + ___ = ___

Ben calls this the "Double-Do-It-To-All-Three" method.

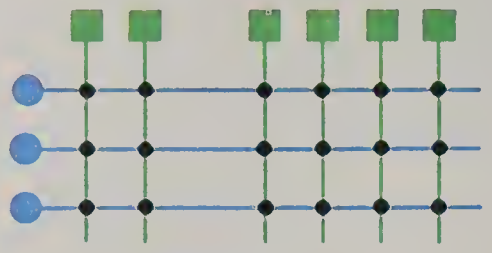
Why does Ben's "Do-It-To-Both" method seem to work?

Our way



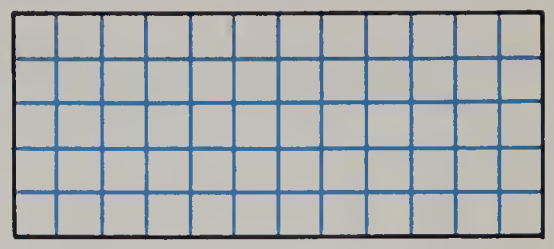
3 X 6 = 18

(A) Ben's "Do-it-to-both" way



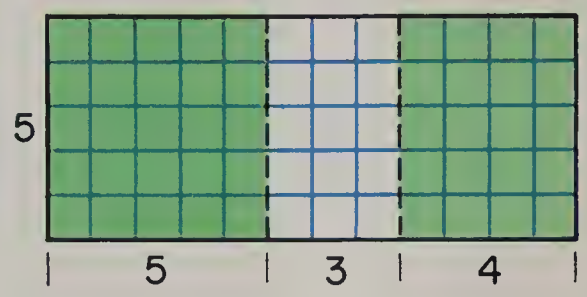
3 X 6 = 3 X (2 + 4) = 6 + 12 = 18

Our way



5 X 12 = ____

(B) Ben's "Do-it-to-all-three" way

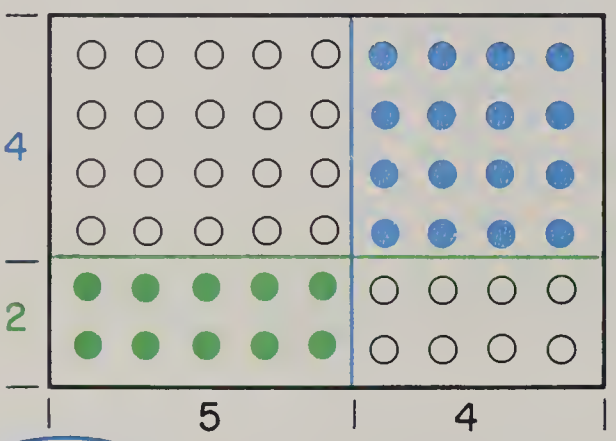


5 X 12 = 5 X (5 + 3 + 4) =

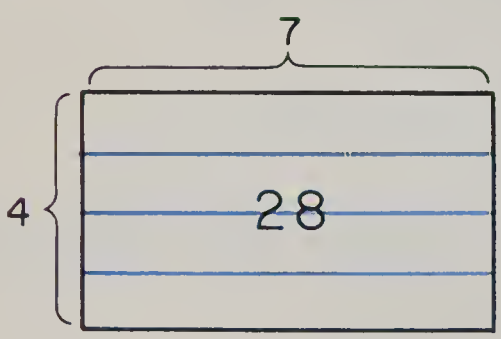
____ + ____ + ____ = ____

Ben's "Double-do-it-to-both" method

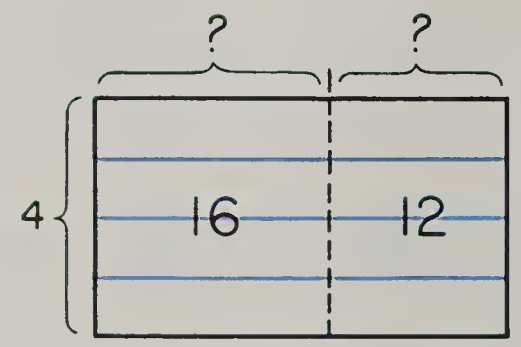
(C)



6 X 9 = (4+2) X (5+4) = ___ + ___ + ___ + ___ = ___



28 ÷ 7 = ____



28 ÷ 4 = (___ + ___) ÷ 4 =

____ + ____ = ____

More practice in "Do-it-to-both"

36 ÷ 4 = (20 + 16) ÷ 4 = ____ + ____ = ____

21 ÷ 3 = (___ + ___) ÷ 3 = ____ + ____ = ____

4 X 8 = 4 X (___ + ___) = ____ + ____ = ____

45 ÷ 5 = _____

40 ÷ 4 = _____

10 X 3 = _____

Practice in "Do-it-to-both"

4 X (3 + 5) = ____ + ____ = ____

(5 + 4) X 3 = ____ + ____ = ____

5 X (5 + 5) = ____ + ____ = ____

3 X (2 + 1) = ____ + ____ = ____

3 X (2 − 1) = ____ − ____ = ____

5 X (5 − 2) = ____ − ____ = ____

(5 − 1) X 4 = ____ − ____ = ____

(5 − 3) X 3 = ____ − ____ = ____

3 X (4 − 4) = ____ − ____ = ____

2 X (1 − 0) = _____

4 X (3 + 2) = _____

(5 + 4) X 4 = _____

Ben said, "Those pictures help me understand it better. I know the method gets me the right answers, but I couldn't explain the reason. Now, I am more sure of it." "I know another way to arrange the work," said Ben.

A. $9 \times 13 = (5+4) \times (5+5+3) = 25 + 25 + 15 + 20 + 20 + 12 = 117$

B $7 \times 8 = (5+2) \times (5+3) = 25 + 15 + 10 + 6 = 56$

C. $3 \times 6 = 3 \times (5+1) = 15 + 3 = 18$

C

3 x 6 = ____

(5 + 1)
x 3
15 + 3 = 18

or

(5 + 1)
x 3
3
+ 15

B

7 x 8 = ____

(5 + 3)
x (5 + 2)
6
10
15
+ 25

A

9 x 13 = ____

(5 + 5 + 3)
x (5 + 4)
12
20
20
15
25
+ 25

Try the "Do-It-To-Both" arrangement used by Ben's sister.

1. 4 x 8 = ____

(5 + 3)
x 4

2. 5 x 6 = ____

3. 4 x 9 = ____

4. 5 x 7 = ____

5. 8 x 8 = ____

(5 + 3)
x (5 + 3)

6. 9 x 7 = ____

7. 6 x 9 = ____

8. 7 x 8 = ____

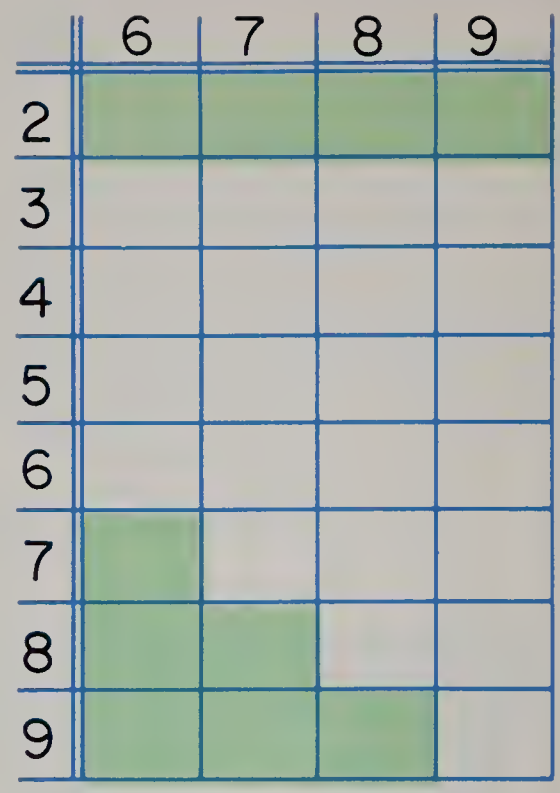
Everyone was interested in Ben's unusual methods. But it took Ben so long to work out the fact that $7 \times 9 = 63$! Everyone ought to memorize that fact!

"Ben," Mrs. Barnes said, "while we learn more about your methods, you ought to learn the multiplication facts between $5 \times 5 = 25$ and $9 \times 9 = 81$."

"Fine," Ben agreed. "Are there many?"

"You said you know that zero times any number is zero; and that multiplying any number by one leaves it unchanged. So, I've put a chart on the board that has a space for all the facts you ought to learn."

"The top row — 2×6, etc. — you know already, because 2×6 is simply $6 + 6 = 12$. You also know that if $6 \times 8 = 48$, then $8 \times 6 = 48$, etc. So, I've shaded areas that won't bother you. That leaves only 22 for you to learn."



"Why don't we learn the facts up to 15×15?", Anne asked. "Of course, we don't absolutely need them the way we work, but wouldn't it save lots of time?"

"It would," Mrs. Barnes admitted. "Let's try. Each day as class starts, Ben can practice the facts he needs to know. Those who know them already can practice remembering such facts as $8 \times 12 = 96$, Here is your first exercise — part for Ben and part for others."

For Ben

3x	7	9	6	8	11	12	15	20	10	13	25	14	50
Product→													

Ruth, Ben's sister, also arranged divisions differently. Here are both ways. Finish the examples. Remember, use no facts beyond $25 \div 5 = 5$.

A. $35 \div 5 = (25 + 10) \div 5 = \_\_\_ + \_\_\_ = \_\_\_$

$5\overline{)35} = 5\overline{)25+10}$ $\quad \frac{5+}{} = \_\_\_$

B. $27 \div 3 = (\_\_\_ + \_\_\_) \div 3 = \_\_\_ + \_\_\_ = \_\_\_$

$3\overline{)27} = 3\overline{)\_\_\_ + \_\_\_}$ $\quad \frac{+}{} = \_\_\_$

C. $32 \div 4 = (\_\_\_ + \_\_\_) \div \_\_\_ = \_\_\_ + \_\_\_ = \_\_\_$

$4\overline{)32} = 4\overline{)\_\_\_ + \_\_\_}$ $\quad \frac{+}{} = \_\_\_$

D. $14 \div 2 = (\_\_\_ + \_\_\_) \div \_\_\_ = \_\_\_ + \_\_\_ = \_\_\_$

$2\overline{)14} = 2\overline{)\_\_\_ + \_\_\_}$ $\quad \frac{+}{} = \_\_\_$

Do-It-To-All-Four

E. $68 \div 4 = (20 + \_\_\_ + \_\_\_ + \_\_\_) \div \_\_\_ = \_\_\_ + \_\_\_ + \_\_\_ + \_\_\_ = \_\_\_$

$4\overline{)68} = 4\overline{)\_\_\_ + \_\_\_ + \_\_\_ + \_\_\_}$ $\quad \frac{+ \quad + \quad +}{} = \_\_\_$

4 x	6	7	9	8	15	11	13	20	14	25	30	12	50
Product													

"After everyone has finished the limbering up exercises, above, I am going to ask Ben to show us how to tackle the problem 56 ÷ 8".

As soon as Ben was finished he put the following on the board.

I

$56 \div 8 = (\ 5 + 3\ ) \overline{)56}$ $5 + 2 = ---$

$25 + 15 = 40$

16

$10 + 6 = 16$

II

$72 \div 9 = (\ 5 + 4\ ) \overline{)72}$ $5 + --- = ---$

$--- + --- = ---$

$--- + --- = ---$

III

$35 \div 7 = (\ 5 + 2\ ) \overline{)35}$ $5 + --- = ---$

$--- + --- = ---$

IV

$63 \div 7 = (--- + ---) \overline{)63}$ $--- + --- = ---$

$--- + --- = ---$

$--- + --- = ---$

V

$36 \div 6 = (--- + ---) \overline{)36}$ $--- + --- = ---$

$--- + --- = ---$

$--- + --- = ---$

Ben explained further: "When I have hard problems, I use 5's as often as I can in both factors."

VI

$168 \div 12 = (\ 5 + 5 + 2\ ) \overline{)168}$ $5 + 5 + --- = ---$

$25 + 25 + 10 = 60$

$--- + --- + --- = ---$

$--- + --- + --- = ---$

VII

Can you use Ben's method?

$169 \div 13 = (--- + --- + ---) \overline{)169}$

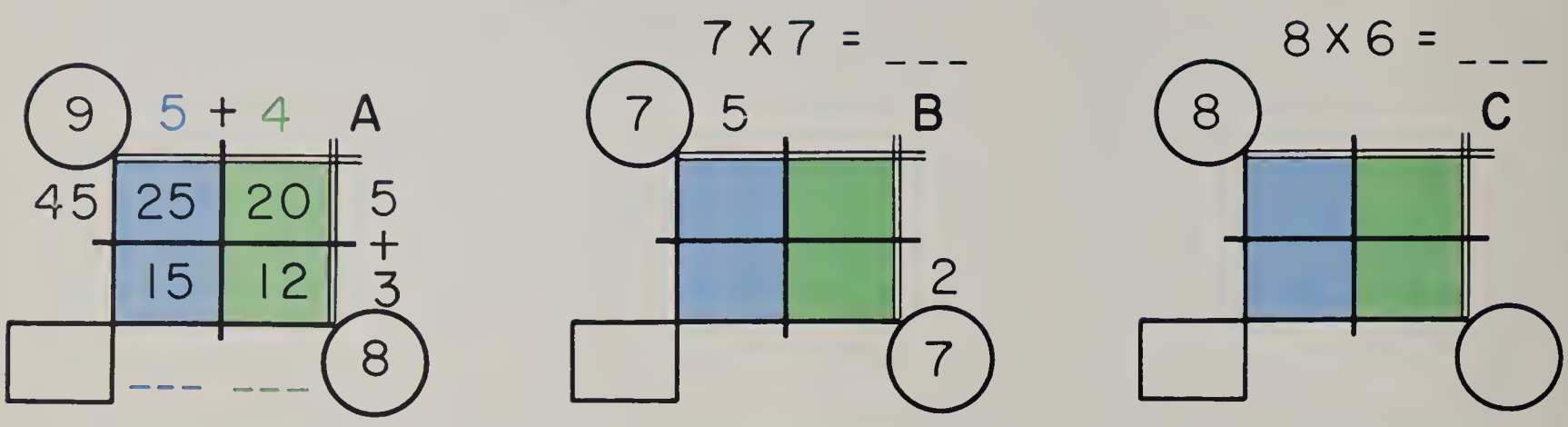
VIII

$108 \div 9 =$

5 x	9	6	8	7	11	13	15	10	12	14	16	25	50
Product													

Anne's way of keeping track of the DOUBLE Do-It-To-Both using the Cross Number Puzzle.

$9 \times 8 = (5 + 4) \times (5 + 3) = 25 + 15 + 20 + \_\_ = \_\_\_\_$

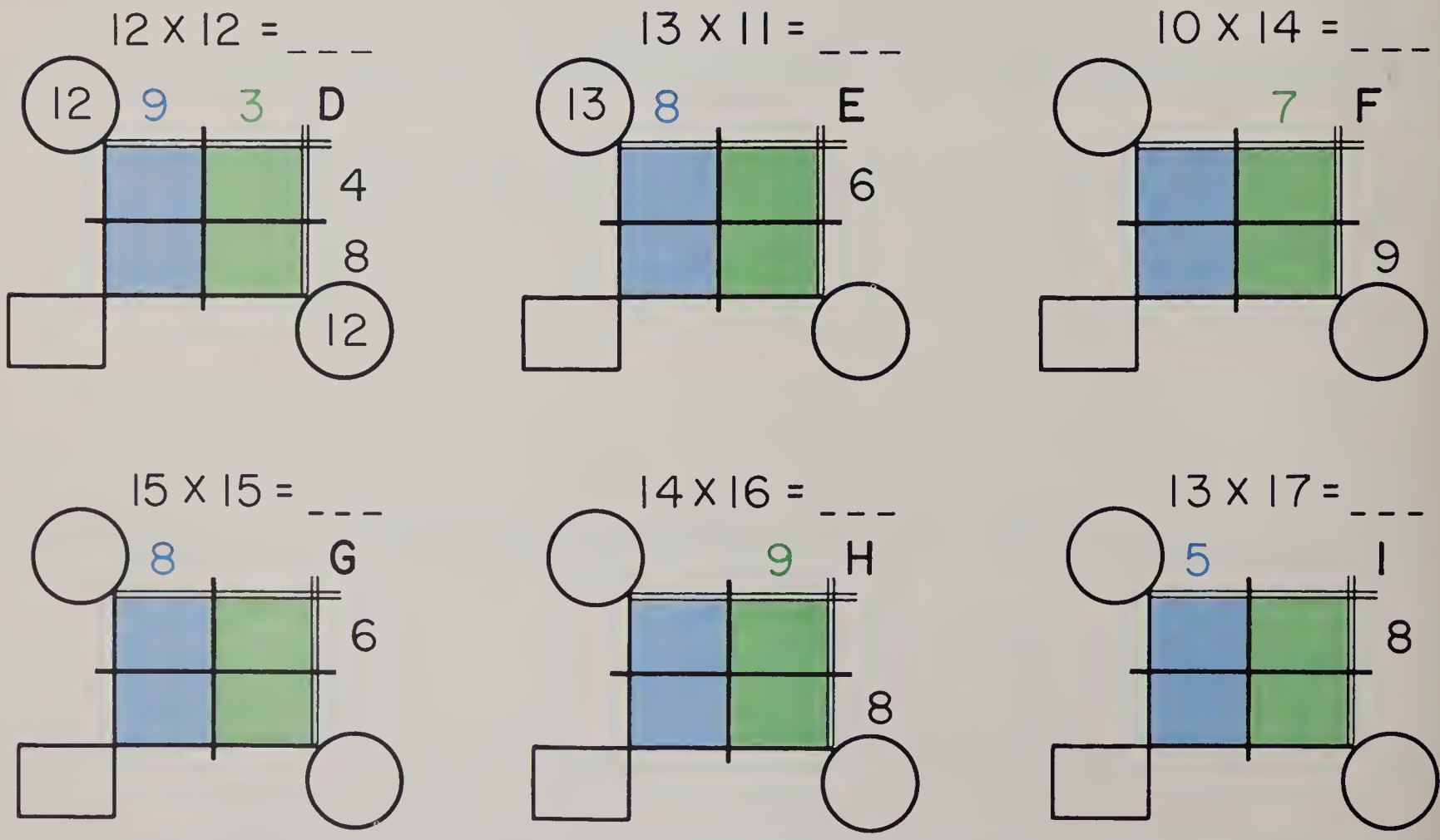


Ben was excited about Anne's idea. He tried many examples.

"How are you coming with your new facts, Ben?", Mrs. Barnes asked.

"I know them all up to $9 \times 9 = 81$. Now I can get to an answer much more quickly."

"Fine," Mrs. Barnes commented. "Then we can all complete the following Cross Number Puzzles using multiplication facts up to $9 \times 9 = 81$."



___ x													
Product	24	32	36	28	40	80	60	100	44	56	48	200	52

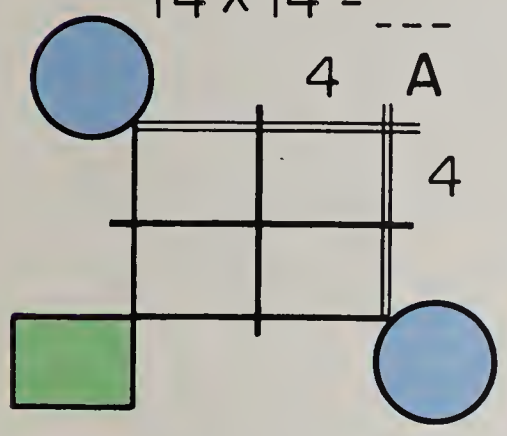
7 x	6	9	7	8	11	50	25	10	16	15	14	20	13
Product													

"Ben's next job," explained Mrs. Barnes, "is an easy and important one. He needs to learn how to multiply by 10."

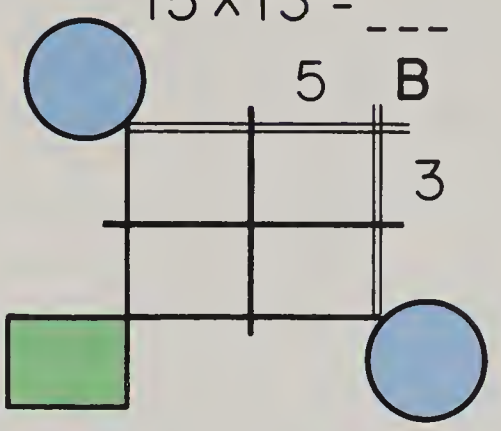
"I know that already," Ben said. "It's so easy I can't understand why it's so important."

"You can find that out for yourself in the following examples," Mrs. Barnes promised.

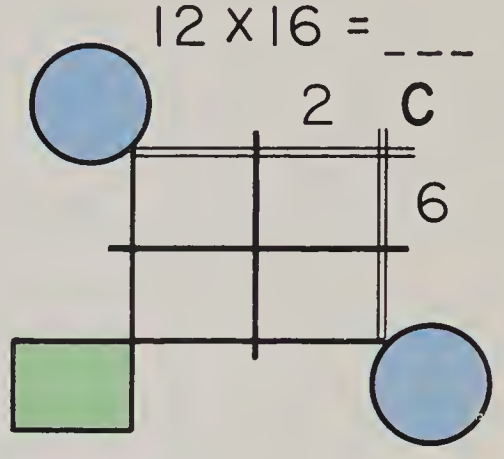
14 X 14 = ___



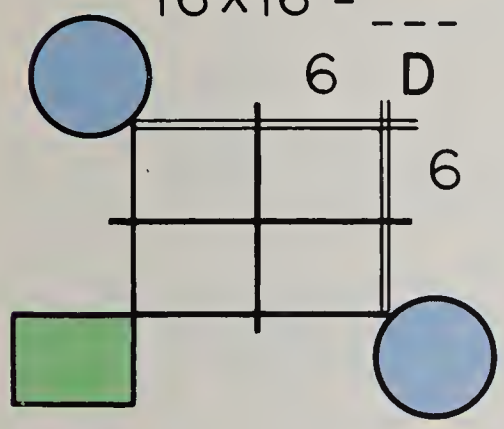
15 X 13 = ___



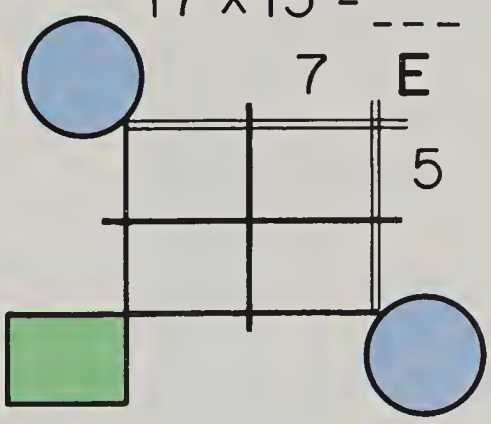
12 X 16 = ___



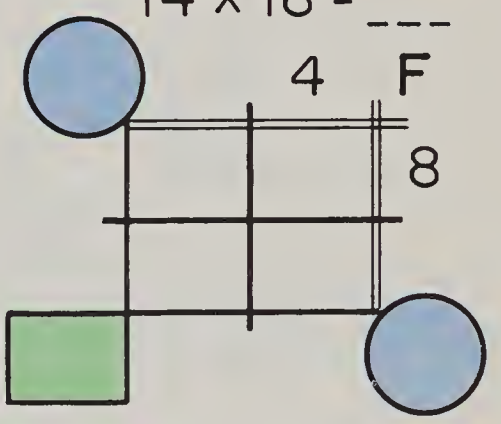
16 X 16 = ___



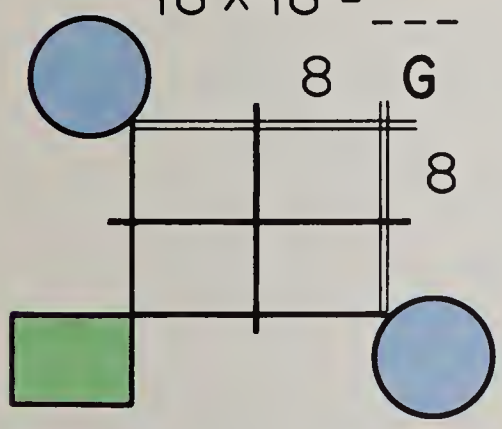
17 X 15 = ___



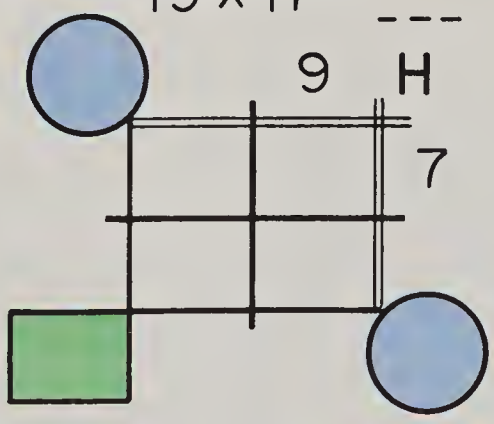
14 X 18 = ___



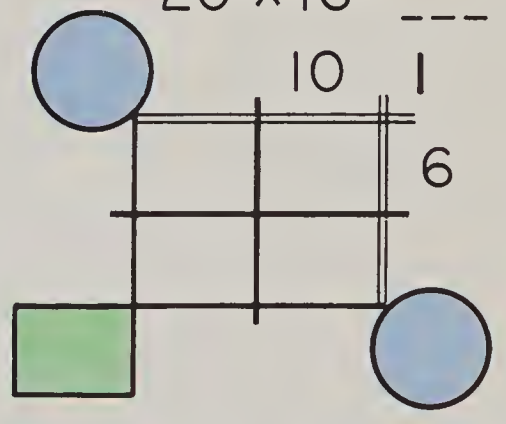
18 X 18 = ___



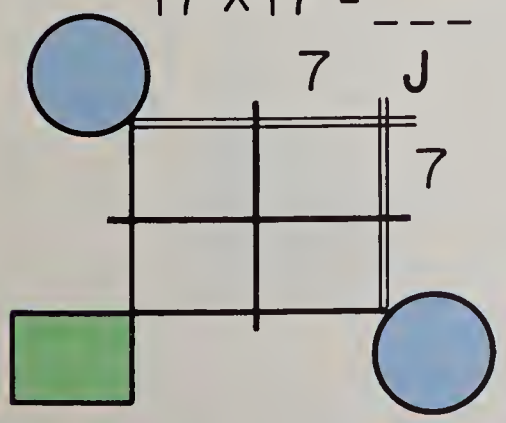
19 X 17 = ___



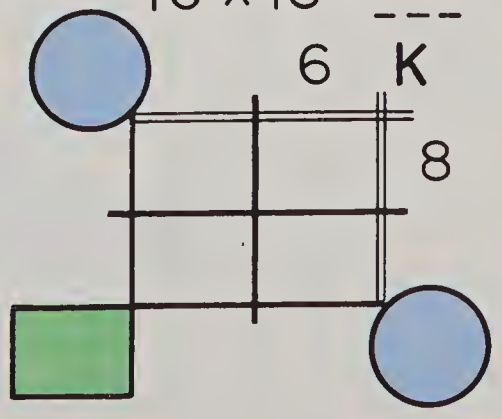
20 X 16 = ___



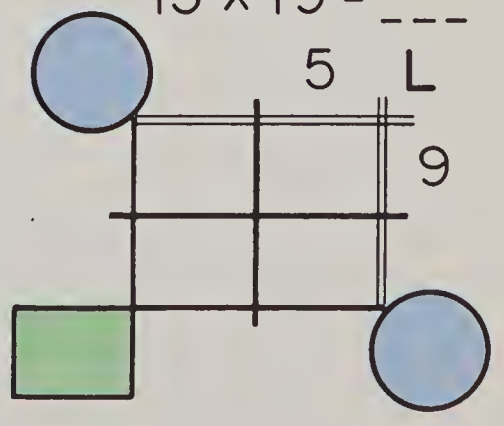
17 X 17 = ___



16 X 18 = ___



15 X 19 = ___



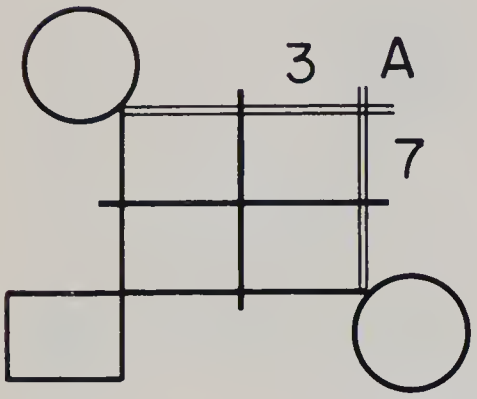
___ X													
Product	48	42	36	54	60	120	78	150	72	96	84	66	90

6 X	6	8	10	12	14	16	7	9	11	13	15	20	25
Product													

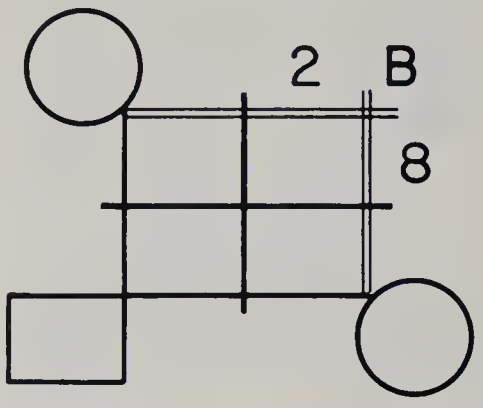
Mrs. Barnes said, "We have found that there are several ways to get a correct answer."

"Today, we are going to complete three Cross Number Puzzles. Then, we will tackle these problems in several ways. We will use Ben's, Ruth's, and Anne's methods,"

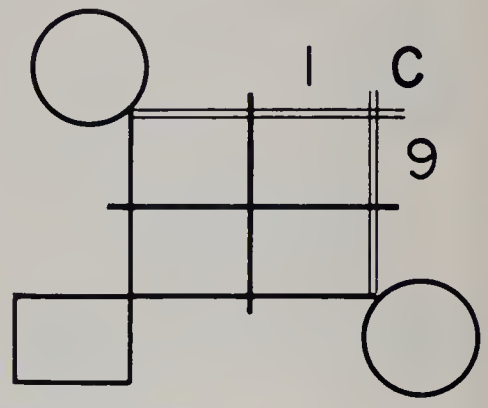
13 X 17 = ____



12 X 18 = ____



11 X 19 = ____



Ben's method: 13 X 17 = (10+3)X(10+7) = ____ + ____ + ____ + ____ = ____

Ruth's method:

(10+3)
X(10+7)
21
70
30
+ 100

Anne's method:

13
X 17
91
130

(7 × 3 = 21. Put down the 1 and remember the 20.)

(7 × 10 = 70. Add the 20 you remember: 70 + 20 = 90.)

(10 × 13 = 130)

Ben: 12 X 18 = (7+5)X(9+9) = ____ + ____ + ____ + ____ = ____

11 X 19 = (3+8)X(9+10) = ____ + ____ + ____ + ____ = ____

Ruth:

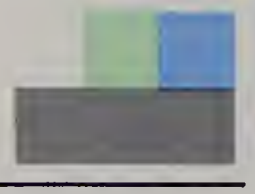
(9+9)
X(4+8)

(10+9)
X(8+3)

Anne:

18
X 12

19
X 11



Compare the different methods

____ X													
Product	56	72	88	48	64	80	104	120	96	112	128	200	400

____ ×	9	11	7		15		25			14	20	13	
				80		96		48	40				64

"In the example, 12 × 16 =____," Mrs. Barnes began, "we have two factors and we must find the product.

"Write the following division problems as problems using multiplication — and rewrite the multiplication problems as division problems."

17 × 13 = ____ or ____ ÷ ____ = ____

240 ÷ 16 = ____ or ____ × ____ = ____

14 × 14 = ____ or

____ = 256 ÷ 16 or

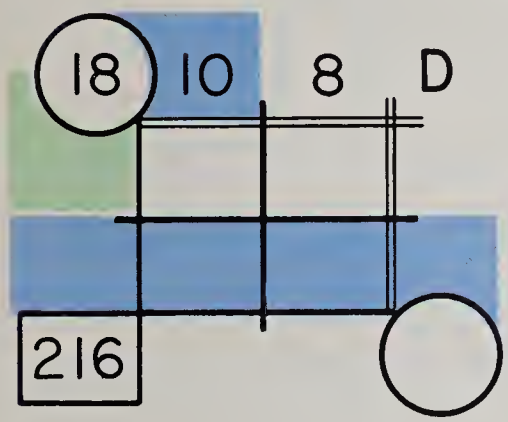
____ = 15 × 15 or ____ = ____ ÷ ____

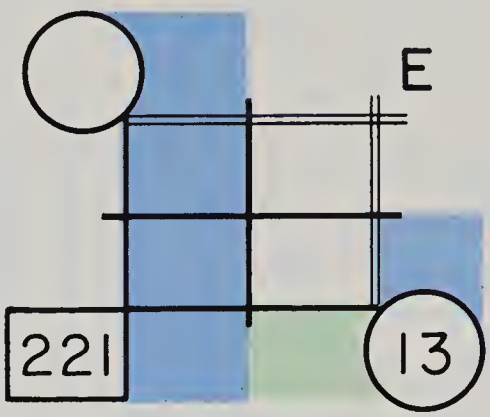
169 = ____ × 13 or

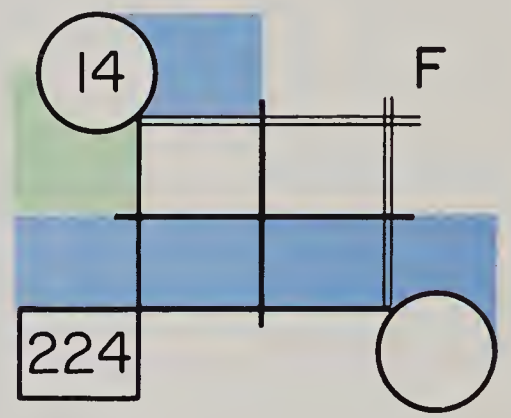
224 ÷ 16 = ____ or

____ = 121 ÷ 11 or

"Let's try to find a missing factor in the Cross Number Puzzle by using Ben's way, Ruth's way, and Anne's way," Mrs. Barnes suggested. "Blue areas are reserved for multiples of 10. Complete them first."







Ben: 18 × ____ = (10 + 8) × (____ + ____) = ____ + ____ + ____ + ____ = 216

____ × 13 = (____ + ____) × (10 + 3) = ____ + ____ + ____ + ____ = 221

14 × ____ = (10 + 4) × (____ + ____) = ____ + ____ + ____ + ____ = 224

Ruth:

(10+8) (+) (10+4)
× (+) × (10+3) × (+)

216 221 224

Anne:

18 × ____ = 216

____ × 13 = 221

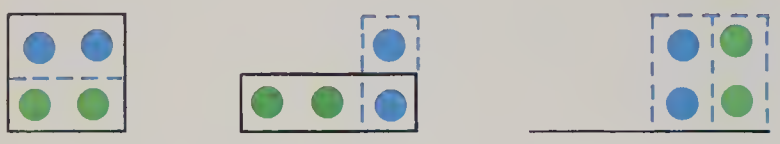
14 × ____ = 224

____ ×	12		6		7		8			9			
		55		25		50		65	75		150	100	70

Keep your eyes open.

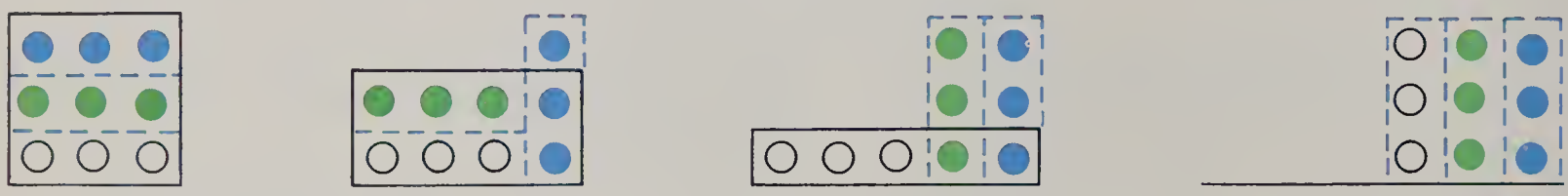
a	b	c	d
2 x 2 = 4 - ___	3 x 3 = 9 - ___	4 x 4 = 16 - ___	5 x 5 = 25 - ___
1 x 3 = 4 - ___	2 x 4 = 9 - ___	3 x 5 = 16 - ___	4 x 6 = 25 - ___
0 x 4 = 4 - ___	1 x 5 = 9 - ___	2 x 6 = 16 - ___	3 x 7 = 25 - ___
	0 x 6 = 9 - ___	1 x 7 = 16 - ___	2 x 8 = 25 - ___
		0 x 8 = 16 - ___	1 x 9 = 25 - ___
			0 x 10 = 25 - ___

a



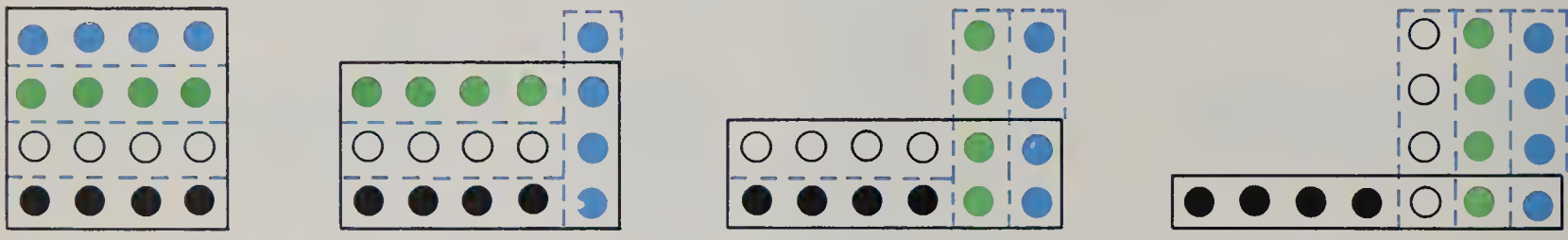
2x2=4-___ 1x3=4-___ 0x4=4-___

b



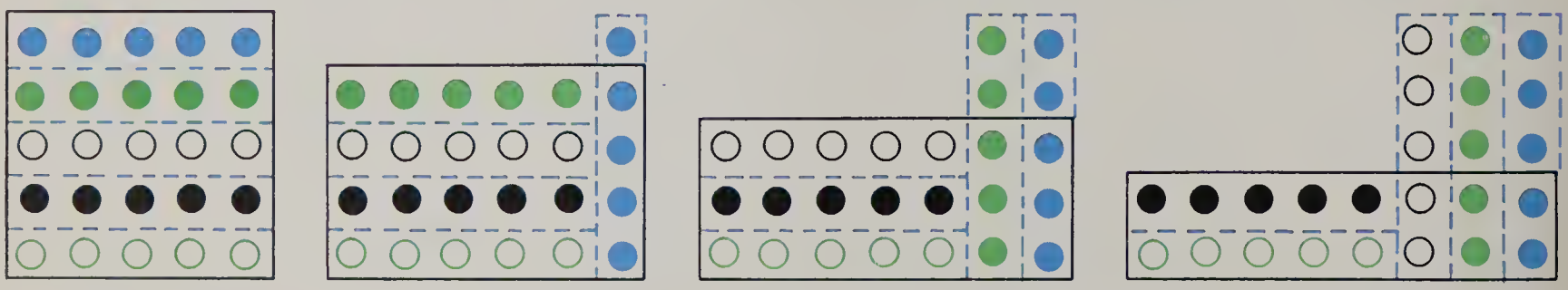
3x3=9-___ 2x4=9-___ 1x5=9-___ 0x6=9-___

c



4x4=16-___ 3x5=16-___ 2x6=16-___ 1x7=16-___

d



5x5=25-___ 4x6=25-___ 3x7=25-___ 2x8=25-___

10 x 10 = 100 - ___ = ___

9 x 11 = 100 - ___ = ___

8 x 12 = 100 - ___ = ___

7 x 13 = 100 - ___ = ___

6 x 14 = 100 - ___ = ___

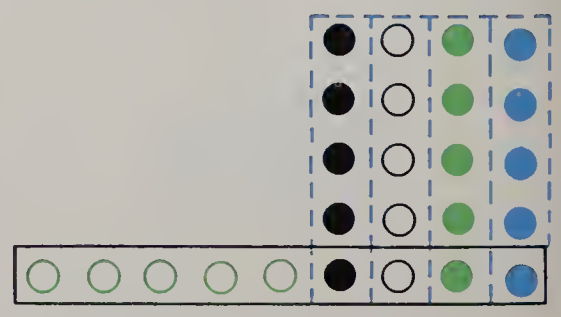
0 x 0 = ___

1 x 1 = ___

2 x 2 = ___

3 x 3 = ___

4 x 4 = ___



1 x 9 = 25 - ___

factor ↓ 3 x factor ↓ 10 = missing product ↓ ___	factor 27 = missing factor ___ x factor 3	factor 5 = product 45 ÷ missing factor ___
product 42 = missing factor ___ x factor 6	product 77 ÷ factor 7 = missing factor ___	factor 12 x missing factor ___ = product 48
product / factor $\frac{28}{7}$ = ___ missing factor	factor 7 ⟌ ___ (35) missing factor / product	factor 8 (7 ⟌ ___) missing product

Multiplication

15 factor	25 factor	___ missing factor	___ missing factor (not 1)
6 factor	___ missing factor	5 factor	___ missing factor (not 1)
___ missing product	75 product	100 product	35 product

f_1 means one factor ; f_2 means the other factor ; p means product.

Make up and complete your own examples without repeating any pair of factors. Draw a circle around the missing term.

2 x 3 = ___	___ ÷ 3 = 2	3 = ___ ÷ 2	___ = ___ ÷ ___	___ = ___ x ___	___ x ___ = ___	p	f_1		
___ = ___ x ___	___ = ___ ÷ ___	___ ÷ ___ = ___	___ = ___ ÷ ___	___ = ___ x ___	___ ÷ ___ = ___	f_1	f_2		
___ ÷ ___ = ___	___ = ___ ÷ ___	___ x ___ = ___	___ = ___ x ___	___ = ___ ÷ ___	___ x ___ = ___	p	f_1		
___ = ___ ÷ ___	___ = ___ x ___	___ ÷ ___ = ___	___ = ___ ÷ ___	___ = ___ x ___	___ ÷ ___ = ___	f_1	p		
___ x ___ = ___	___ = ___ ÷ ___	___ = ___ x ___	___ = ___ x ___	___ = ___ ÷ ___	___ = ___ x ___	f_1	f_2		
___ = ___ ÷ ___	___ = ___ x ___	___ ÷ ___ = ___	___ = ___ ÷ ___	___ = ___ x ___	___ ÷ ___ = ___	p	f_1		
___ ÷ ___ = ___	___ = ___ ÷ ___	___ x ___ = ___	___ = ___ ÷ ___	___ = ___ x ___	___ ÷ ___ = ___	p	f_1		
⟌	⟌	⟌	⟌	⟌	⟌	p	f_1		

Given Terms		Find
f_1	f_2	
p	f_1	
f_1	f_2	
p	f_1	
f_1	p	
f_1	f_2	
p	f_1	
p	f_1	

Sums →	8	16	14	13	11	17	13	10	11	9
Two Addends ↑	3	7	12	9	8					
	5	9	2							
	15	63	24	36	24	30	30	24	30	18

↓ Two Factors

← Products

ADDENDS — SUMS — FACTORS — PRODUCTS

Let's agree on the meaning of four simple words: ADDEND, SUM, FACTOR, and PRODUCT.

Since addition and subtraction are just different ways of talking about the same idea, we will describe the operations with the words 'addends' and a 'sum.' Multiplication and division will be described in terms of 'factors and a product'.

Addition and Subtraction:

one addend	other addend	sum	one addend	other addend	missing sum	one addend	missing addend	sum
3	+ 4	= 7	8	+ 12	= ___	15	+ ___	= 21

sum	addend	missing sum	addend	sum	missing addend	missing sum	addend	addend
16	− 9	= ___	13	= 25	− ___	___	− 16	= 6

Addition:

18	addend	19	addend
6	addend	___	missing addend
___	missing sum	26	sum
___	missing addend	33	addend
13	addend	21	addend
40	sum	___	missing sum

Subtraction:

33	sum	___	missing sum
8	addend	6	addend
___	missing addend	25	addend
24	sum	29	sum
___	missing addend	11	addend
14	addend	___	missing addend

a_1 means one addend; a_2 means another addend; s means sum

		Given Terms		Find
1.	13 + 7 = ___	1.			
2.	25 = ___ + 15	2.			
3.	25 − ___ = 17	3.			
4.	___ − 9 = 30	4.			
5.	24 + ___ = 44	5.			
6.	0 = ___ − 18	6.			

Addition:

I	II	III
7	18	___
9	___	14
___	24	14

Subtraction:

IV	V	VI
20	13	___
9	___	8
___	7	22

	Given Terms		Find
I			
II			
III			
IV			
V			
VI			

Sums →	7	13	14	11	12	17		4				10
Addends ↑ and Factors ↓ (other than "1")	3	7	9	8								
	4	6	5		7	8						
Products →	12	42		24	35		21		27	25	33	24

You have already worked out all the facts printed in green.
Have you learned a secret that makes the rest very easy?

15 x 15 = 225 − 0 = ___	16 x 16 = ___ − ___ = ___	17 x 17 = ___ − ___ = ___
14 x 16 = 225 − 1 = ___	15 x 17 = ___ − ___ = ___	16 x 18 = ___ − ___ = ___
13 x 17 = 225 − ___ = ___	14 x 18 = ___ − ___ = ___	15 x 19 = ___ − ___ = ___
12 x 18 = ___ − ___ = ___	13 x 19 = ___ − ___ = ___	14 x 20 = ___ − ___ = ___
11 x 19 = ___ − ___ = ___	12 x 20 = ___ − ___ = ___	13 x 21 = ___ − ___ = ___
10 x 20 = ___ − ___ = ___	11 x 21 = ___ − ___ = ___	12 x 22 = ___ − ___ = ___

19 x 21 = 20 x 20 − 1 = ___	3 x 7 = 5 x 5 − ___ = ___
11 x 9 = 10 x 10 − ___ = ___	12 x 8 = ___ x ___ − ___ = ___
13 x 11 = ___ x ___ − ___ = ___	18 x 22 = ___ x ___ − ___ = ___
31 x 29 = ___ x ___ − ___ = ___	14 x 10 = 12 x ___ − ___ = ___
7 x 13 = 10 x 10 − ___ = ___	6 x 14 = 10 x ___ − ___ = ___
2 x 8 = 5 x ___ − ___ = ___	24 x 16 = ___ x ___ − ___ = ___
17 x 23 = ___ x ___ − ___ = ___	26 x 34 = ___ x ___ − ___ = ___
33 x 27 = ___ x ___ − ___ = ___	8 x 16 = 12 x ___ − ___ = ___

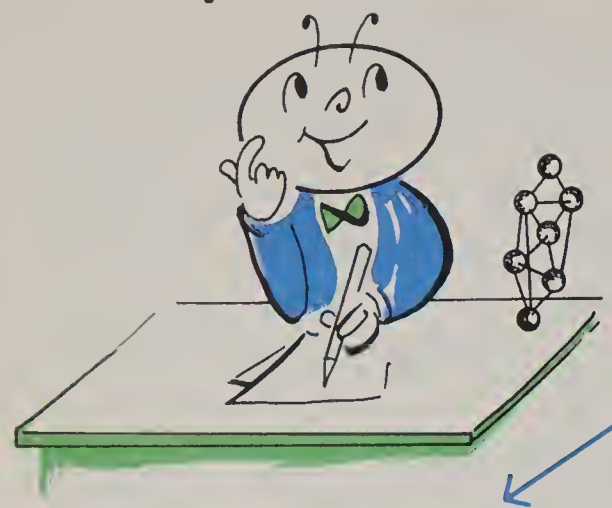
If you know the following facts, there are many others you can do in your head.

25 x 25 = 625 35 x 35 = 1225 45 x 45 = 2025 55 x 55 = 3025

(In your spare time, study the underlining in color. There is a secret hidden there. When you discover the secret, test it on 65 × 65.)

24 x 26 = 25 x 25 − ___ = ___	53 x 57 = ___ x ___ − ___ = ___
36 x 34 = 35 x ___ − ___ = ___	22 x 28 = ___ x ___ − ___ = ___
44 x 46 = 45 x ___ − ___ = ___	21 x 29 = ___ x ___ − ___ = ___
23 x 27 = ___ x ___ − ___ = ___	20 x 30 = 25 x 25 − ___ = ___
47 x 43 = ___ x ___ − ___ = ___	38 x 32 = ___ x ___ − ___ = ___
33 x 37 = ___ x ___ − ___ = ___	39 x 31 = ___ x ___ − ___ = ___
54 x 56 = ___ x ___ − ___ = ___	40 x 30 = 35 x 35 − ___ = ___

MAGIC SQUARES



A.

15	1	11
5	9	13
7	17	3

B. A + 26

41	27	
31		

C. A X 5

75	5	
25		

D. A X 6

90	6	
30		

E. B + C

F. D − C

G. D + 18

H. E ÷ 4

I. E − G

J. G ÷ 4

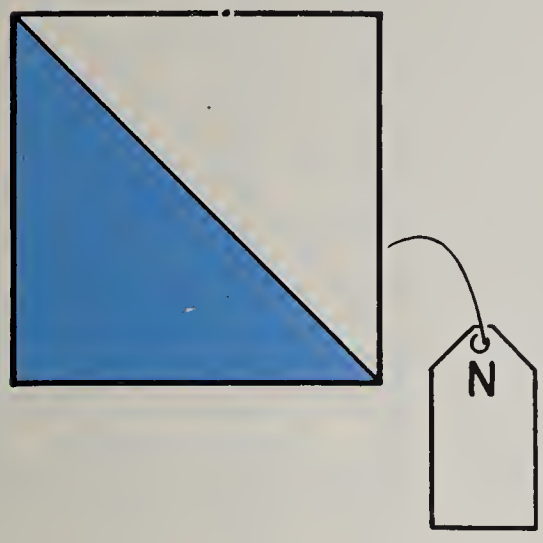
K. E + H

L. G + J

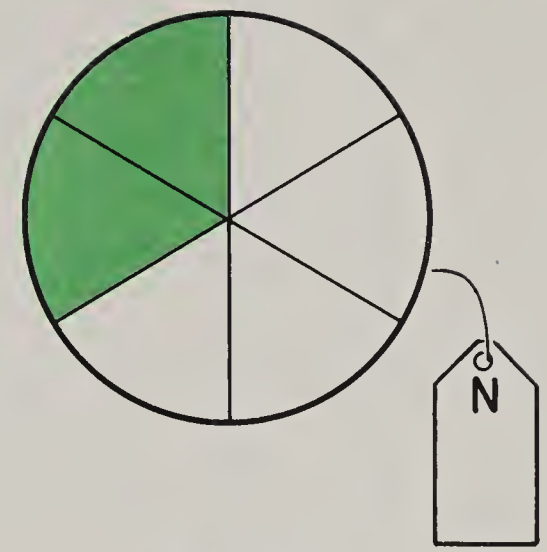
M. F X I

"N" on the tags means the number suggested by the whole shape.

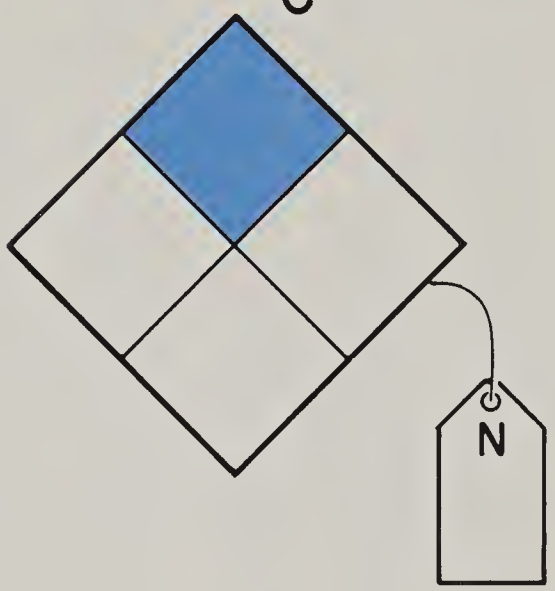
A



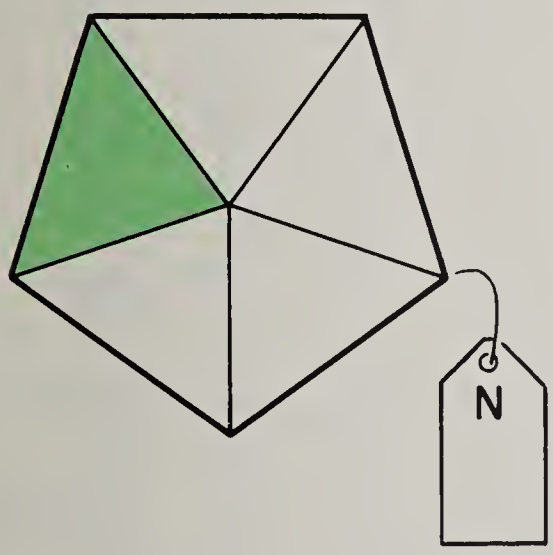
B



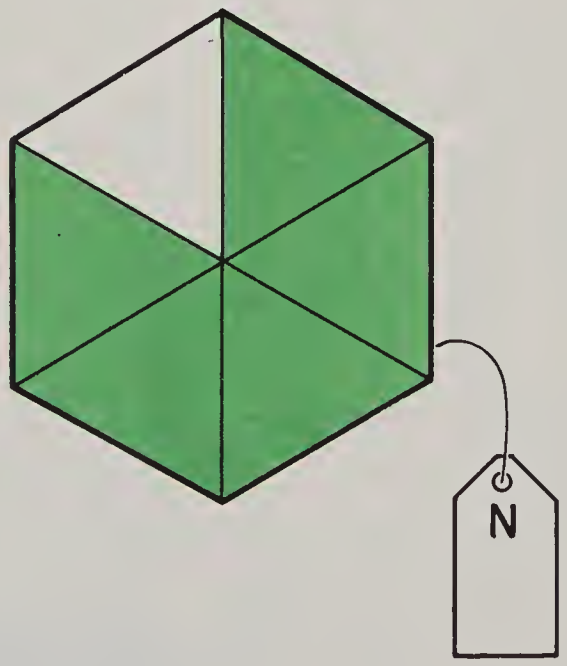
C



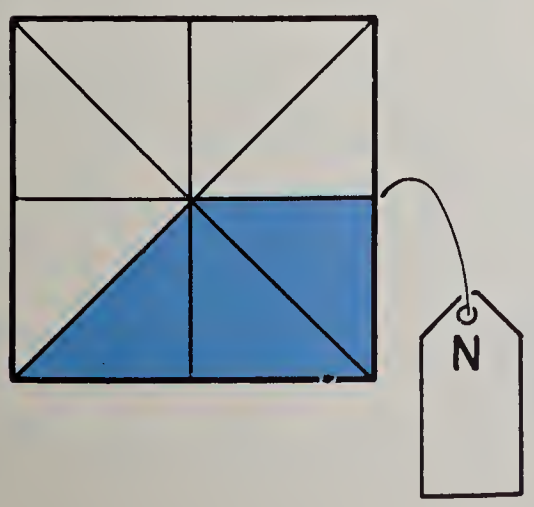
D



E



F

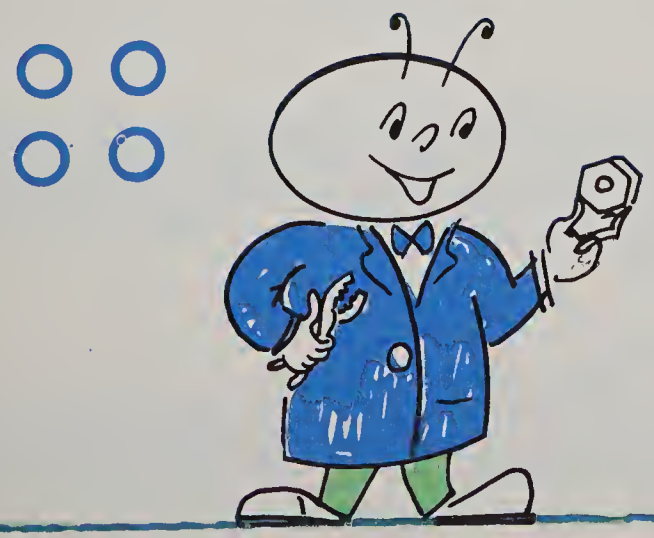


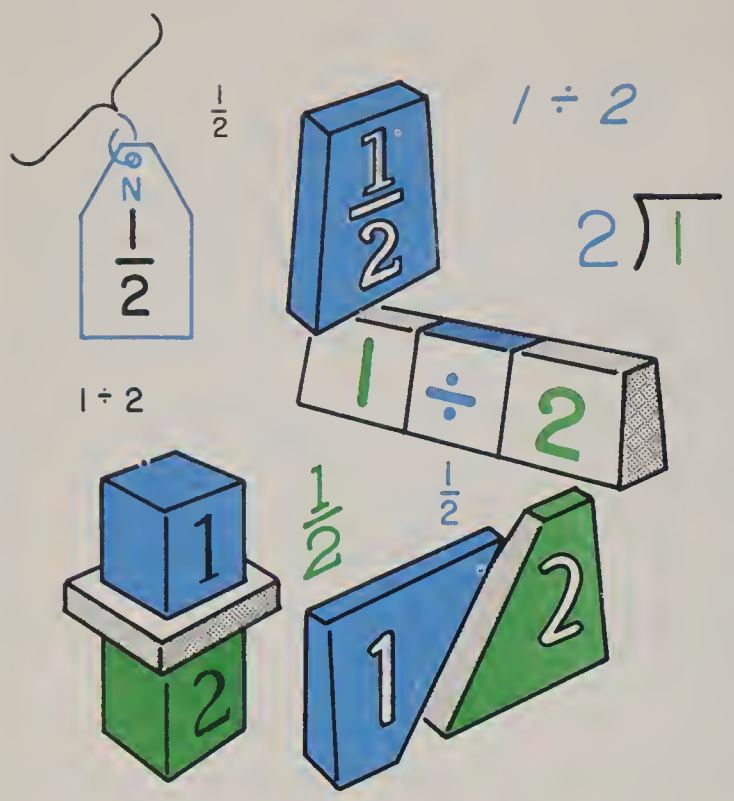
	Shaded	Un-shaded	Tag
A	3		
B	3		
C	3		
D	3		
E		3	
F	3		
A			8
B			12
C			20
D			20
B		20	
C		18	
E	50		
F		15	
A			100

Extend the SERIES

1st	2nd	3rd	4th	5th	6th		10th	11th		14th	15th	16th		20th
1	2	3												
2	4	6												
3	4	5												
1	3	5	7											
1	4	7												
0	5	10												
2	7	12												
20	19	18	17											
3	6	9												
3	7	11												
5	8	11												
75	72	69												
1	2	4	7	11	16									
1	4	9	16											
1	2	4	8											
1	1	2	3	5										
1	3	9	27											

	A	B	C	D	E	F	G	H	I	J
j										81
i			16						64	
h				21				49		
g			12							
f					20					
e	0		8			20				
d				9		15				
c				6	8					18
b		1					6			
a							0			





Is a FRACTION a NUMBER?

"My brother Bill said a fraction is a number," said Bobbie. "My sister said a fraction is a pair of numbers. My Aunt thought a fraction was either a numeral or a pair of numerals."

"My sister said that fractions and whole numbers belong to two very different number systems. Bill said she was wrong because you can mix fractions and whole numbers, so they can't be so very different. Now, I'm all mixed up. Who is right?"

Mrs. Barnes asked the class if they could help Bobbie. Not a hand went up.

"Well," Mrs. Barnes said, "even mathematicians disagree about some of those questions. Let's not tackle those problems yet. Rather, let's look a little deeper into fractions."

"First, how do we use the idea of fractions in every day life? Here are some statements that mention fractions. What do they mean?"

a. My uncle gave me a half-dollar.

b. How many eggs are there in a half-dozen eggs?

c. Bill traded me one out of every two of his 36 marbles for my model airplane.

d. I paid 25c for a book. The clerk told me that was one-half of the regular price.

e. When I asked the clerk for a pint of cream, he asked me: "Do you want it in a pint carton or two half-pint cartons?"

f. At the carnival the ticket seller shouted: "Who's next to buy a ticket — only one dime, a mere two nickels, two-fifths of a quarter; yes sir, just one-tenth of a dollar. Who's next?"

g. Jim ran one-third of the way to the store and walked the other four blocks.

h. The swimming pool was 60 feet long. Alice swam two-thirds of the length before she got tired.

i. Pencils were 3 for 10c. I got nine of them.

j. Five girls made two dollars profit on a bake sale. They divided the money evenly.

k. I'll be ready in a quarter of an hour.

l. Because of trouble that developed, they stopped the countdown a half-minute before blast-off time.

m. Uncle Pete gave me and Alice a half-dollar and asked us to divide it evenly between us.

n. Lois got four out of every five problems right. She got 20 right.

o. The plane was three-quarters of an hour late.

Sums →	12	15			12		18				
Two Addends ↑	9	7	8	5			10				↓ Two Factors
	3	8	8	7	4	8					
	27		64			48		45	81	49	33 ← Products

"Let's take our study of fractions further," Mrs. Barnes suggested. "We know a lot about whole numbers. Let's see if fractions behave in a similar way. Try these tests. They may shed light on the problem."

Mrs. Barnes' TEST ONE: A group of whole numbers can be arranged in a definite order, from smallest to largest. If "<" means "Less than," we can say that the group consisting of 9, 3, and 15 can be arranged so that $3 < 9 < 15$ is a true statement. Can a group of fractions be similarly arranged?

(whole numbers)

3 , 7 , 1 , 6 , 5

1 < 3 < _ < _ < _

$\frac{8}{2}, \frac{6}{2}, \frac{2}{2}, \frac{5}{2}, \frac{1}{2}$

$\frac{1}{2}$ < _ < _ < _ < _

$\frac{6}{3}, \frac{9}{3}, \frac{3}{3}, \frac{1}{3}, \frac{2}{3}$

_ < _ < _ < _ < _

$\frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{4}, \frac{1}{8}$

_ < _ < _ < _ < _

$\frac{3}{3}, \frac{3}{1}, \frac{3}{2}, \frac{3}{6}, \frac{3}{4}$

_ < _ < _ < _ < _

$\frac{2}{8}, \frac{2}{5}, \frac{2}{2}, \frac{2}{1}, \frac{2}{4}$

_ < _ < _ < _ < _

$\frac{6}{3}, \frac{8}{2}, \frac{4}{4}, \frac{1}{2}, \frac{3}{4}$

_ < _ < _ < _ < _

$\frac{12}{4}, \frac{1}{8}, \frac{2}{3}, \frac{3}{2}, \frac{8}{1}$

_ < _ < _ < _ < _

$\frac{3}{2}, \frac{1}{2}, \frac{9}{4}, \frac{8}{4}, \frac{4}{5}$

_ < _ < _ < _ < _

$\frac{5}{5}, \frac{1}{3}, \frac{6}{8}, \frac{2}{4}, \frac{4}{6}$

_ < _ < _ < _ < _

$\frac{1}{6}, \frac{3}{4}, \frac{1}{2}, \frac{2}{3}, \frac{1}{8}$

_ < _ < _ < _ < _

$\frac{3}{6}, \frac{8}{9}, \frac{1}{5}, \frac{4}{7}, \frac{6}{8}$

_ < _ < _ < _ < _

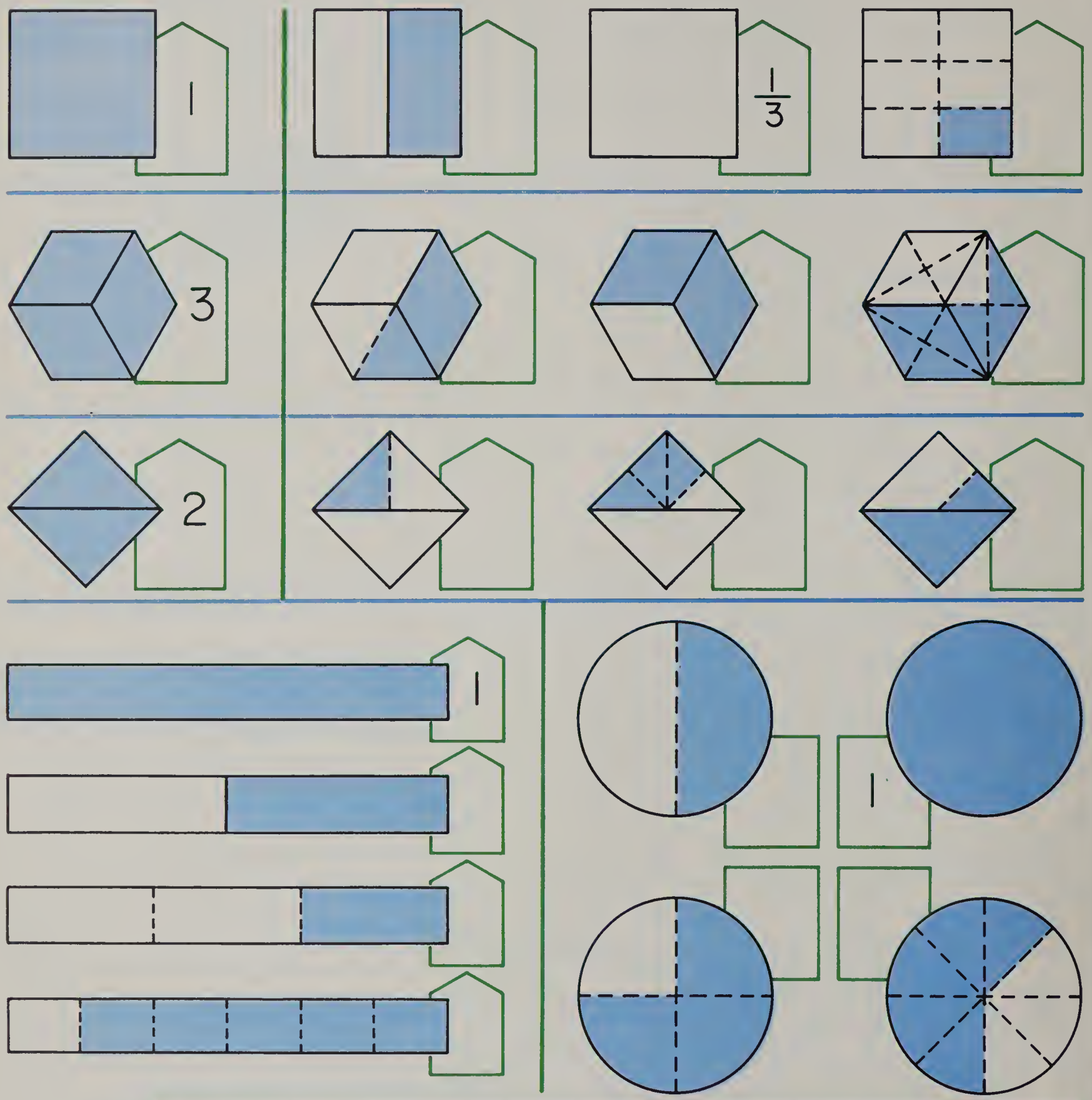
Brain Buster

$\frac{1}{6}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{10}, \frac{1}{3}, \frac{5}{6}, \frac{1}{4}$ → _ < _ < _ < _ < _ < _ < _ < _

Sums	16	48	20		48			100	28	44	90	
Two Addends ↑	12		15	18	42		25	75	21		80	Products
	4	8				5				11		One Factor (not 1)
	3	5	3	9		7						Other Factor

Mrs. Barnes' TEST TWO: Can you draw pictures that suggest the idea for fractions? Can you do it so the pictures include the idea of whole numbers?

If this picture is labeled to suggest a whole number — can you label the rest of the pictures and make sense?



Sums →	16	16	16	16	16	16	5	25		4			20
Addends ↑ Factors ↓ (not 1)													
Products	48	63	39	28	60	55		100	77		121	49	100

Mrs. Barnes' TEST THREE: Are there any fractions that are other names for whole numbers?

Bill volunteered first. "Two half-dollars is another name for one whole dollar. This suggests the idea that $\frac{2}{2}$ might be another name for the whole number, 1."

"Eight quarters is another name for two dollars," Betty added. "I think $\frac{8}{4}$ is another name for 2."

"Ten half-gallons of milk is the same amount as five gallons of milk," said Jane. And it makes sense to me to say that $\frac{10}{2}$ is another name for 5."

Mrs. Barnes began writing on the board. "Which fraction names whole numbers?"

Draw a circle around other names for whole numbers.

$\frac{2}{1}$ $\frac{1}{2}$ $\frac{50}{100}$ $\frac{49}{7}$ $\frac{81}{9}$ $\frac{4}{1}$ $\frac{16}{8}$ $\frac{8}{24}$ $\frac{100}{50}$ $\frac{5}{10}$ $\frac{37}{3}$ $\frac{20}{10}$ $\frac{4}{5}$

$\frac{9}{27}$ $\frac{27}{9}$ $\frac{100}{25}$ $\frac{5}{1}$ $\frac{19}{9}$ $\frac{8}{16}$ $\frac{3}{6}$ $\frac{33}{11}$ $\frac{12}{9}$ $\frac{3}{9}$ $\frac{18}{4}$ $\frac{3}{1}$ $\frac{64}{7}$

$\frac{30}{10}$ $\frac{6}{3}$ $\frac{51}{3}$

$\frac{12}{1}$ $\frac{12}{2}$ $\frac{12}{3}$ $\frac{12}{4}$ $\frac{12}{5}$ $\frac{12}{6}$ $\frac{12}{7}$ $\frac{12}{8}$ $\frac{12}{9}$ $\frac{12}{10}$ $\frac{12}{11}$ $\frac{12}{12}$ $\frac{12}{13}$ $\frac{12}{14}$ $\frac{12}{15}$ $\frac{12}{16}$ $\frac{12}{17}$ $\frac{12}{18}$

Mrs. Barnes' TEST FOUR: Is there at least one fraction to name each whole number?

In each of the boxes below, write fractions that are other names for the whole numbers in the blue boxes.

2	5	3	1	4	0

Make the following into TRUE STATEMENTS:

< means less than, > means greater than, = means equal.

$\frac{2}{1} \blacksquare \frac{8}{2}$ $\frac{9}{3} \blacksquare \frac{12}{4}$ $\frac{21}{7} \blacksquare \frac{14}{7}$ $\frac{50}{10} \blacksquare \frac{36}{9}$ $\frac{1}{1} \blacksquare \frac{12}{12}$ $\frac{6}{2} \blacksquare \frac{28}{7}$

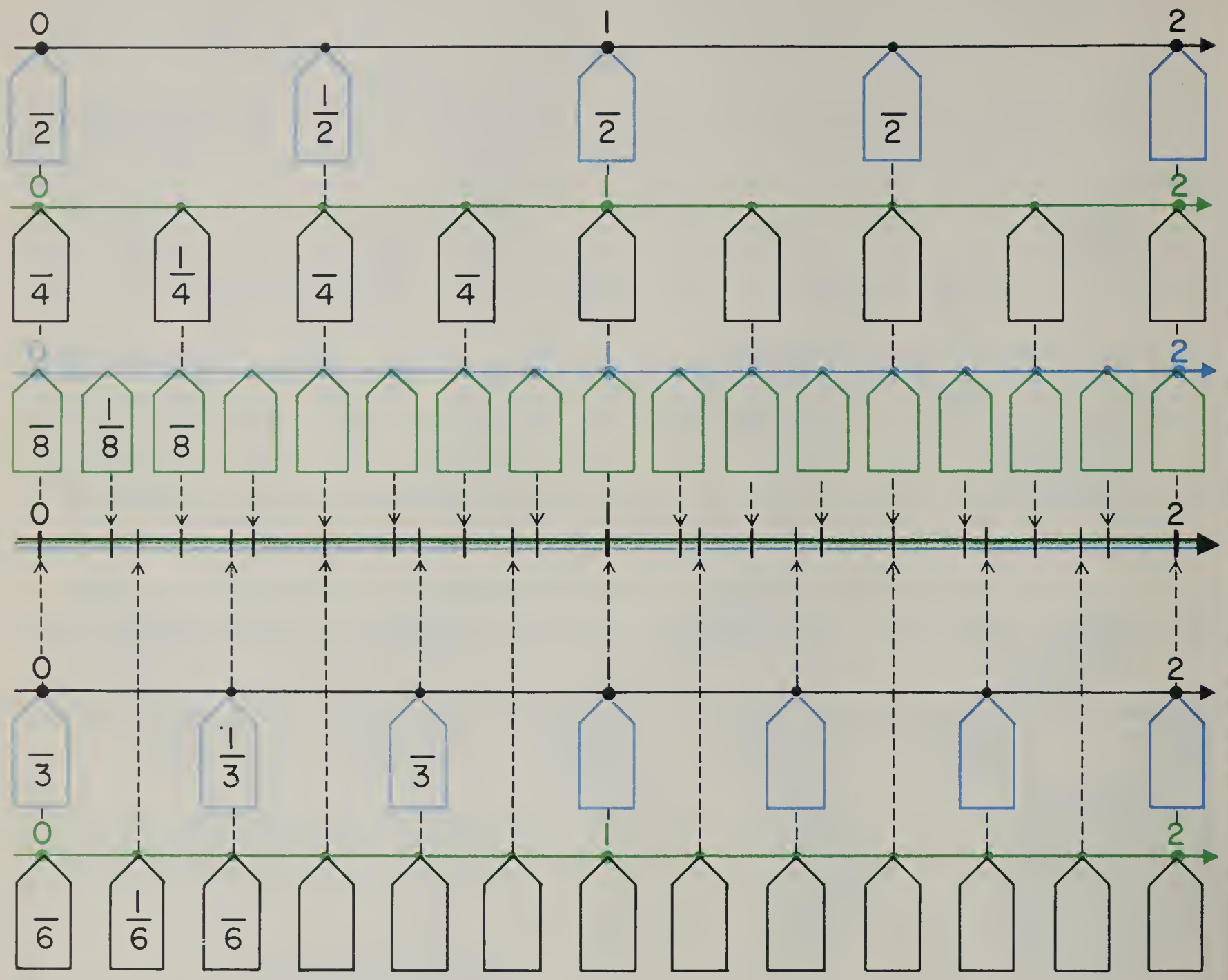
$\frac{55}{11} \blacksquare \frac{12}{3}$ $\frac{10}{10} \blacksquare \frac{1}{1}$ $\frac{18}{2} \blacksquare \frac{24}{3}$ $\frac{1}{1} \blacksquare \frac{0}{7}$ $\frac{72}{9} \blacksquare \frac{63}{7}$ $\frac{100}{100} \blacksquare \frac{77}{77}$

			12		9			44			0
Factor →	8			25		9			21	6	
Factor →		8			6	9	3	6	4	0	
Product →	24	96	45	125			90				0

← Addend ← Sum ← Addend

Mrs. Barnes' TEST FIVE: Can we use fractions as labels for points on the number line?

Fill in the tags below so that they make sense as labels.



Using = , > or < , make the following true.

$\frac{1}{2} \square \frac{3}{4}$	$\frac{2}{6} \square \frac{1}{3}$	$\frac{7}{6} \square \frac{8}{8}$
$\frac{1}{4} \square \frac{2}{8}$	$\frac{4}{6} \square \frac{3}{4}$	$\frac{12}{6} \square \frac{16}{8}$
$\frac{1}{6} \square \frac{1}{8}$	$\frac{6}{6} \square \frac{4}{4}$	$\frac{9}{8} \square \frac{8}{6}$

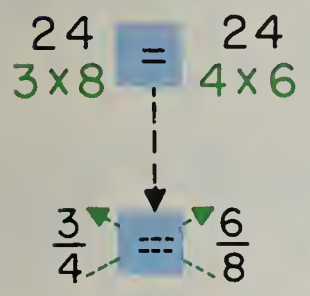
$\frac{8}{6} \square \frac{11}{8}$	$\frac{5}{3} \square \frac{3}{5}$	$\frac{7}{8} \square \frac{5}{6}$
$\frac{6}{8} \square \frac{4}{6}$	$\frac{7}{4} \square \frac{10}{6}$	$\frac{7}{4} \square \frac{14}{8}$
$\frac{1}{4} \square \frac{2}{6}$	$\frac{3}{2} \square \frac{9}{6}$	$\frac{3}{6} \square \frac{2}{4}$

12x →	2		5		9	10	8	18	11	7	30	50	
Product		36		48									240

Ben's Rule:

"I noticed something on the last page that may always be true. If two fractions name the same point on the number line, then I can cross-multiply and get the same number." Mrs. Barnes asked Ben to give the class some examples.

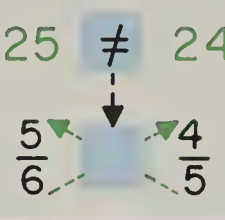
Example I



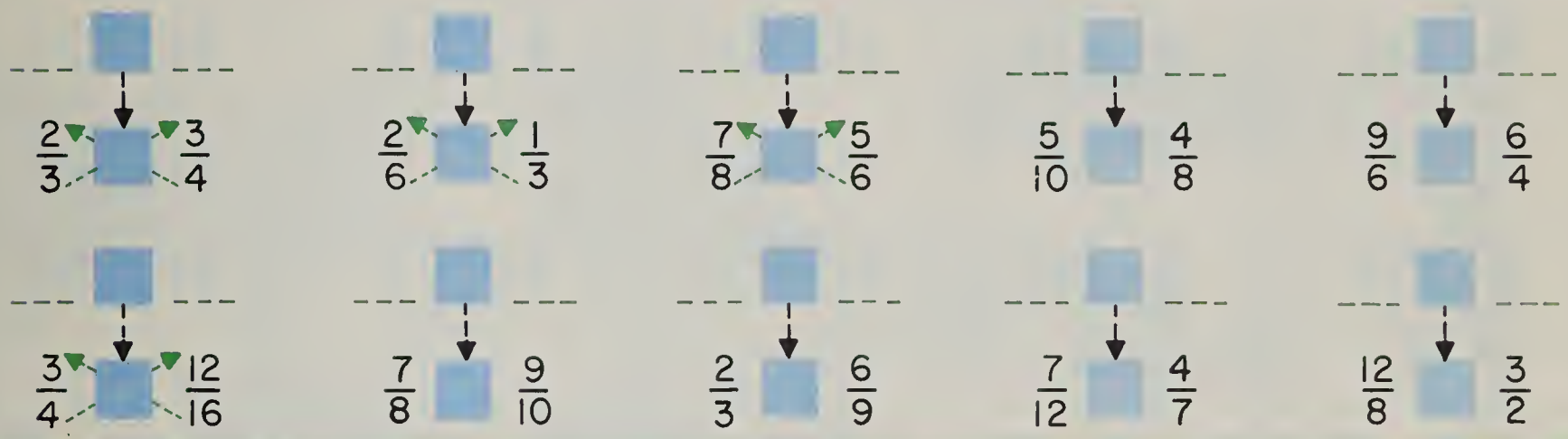
Before Ben went on Mrs. Barnes explained that "≠" means "does not equal".

Example II

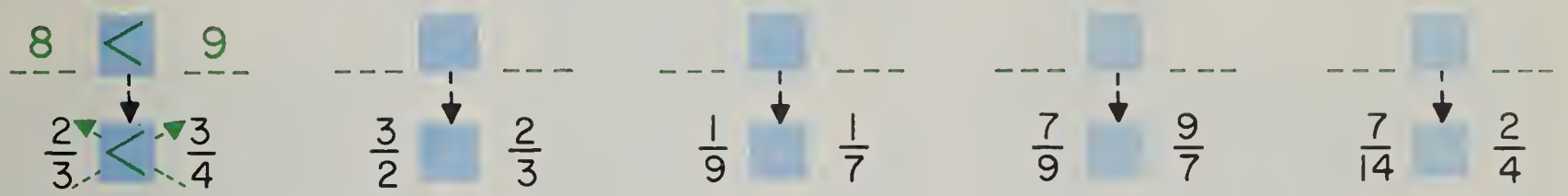
Example III

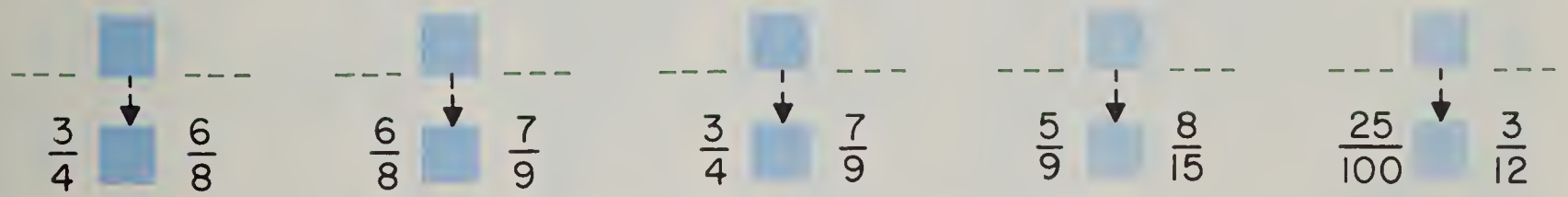


Let's test Ben's rule with more examples.

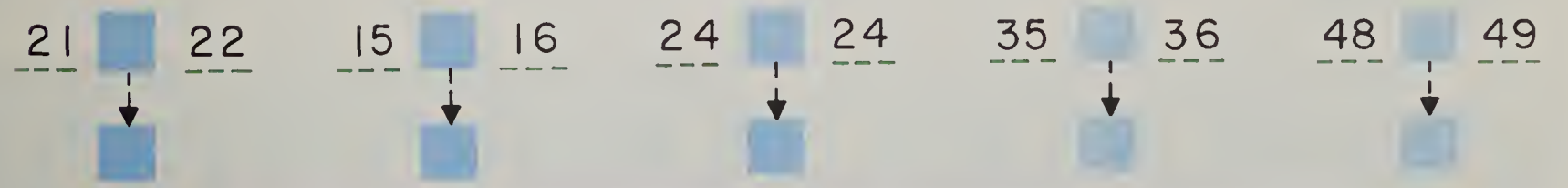


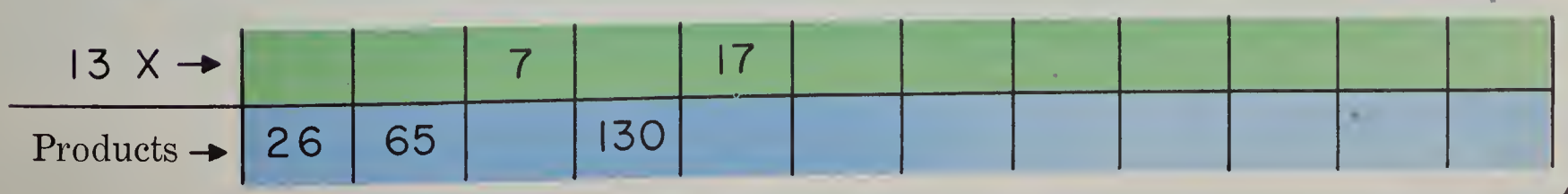
Anne's Rule: "I think Ben could go further," Anne suggested. "He could have put signs for 'greater than' and 'less than' — '>' and '<' — in the lower blue box. Which ever sign makes that statement below true, will also make the statement above it true . . . I think." "Is that right?" Mary wanted to know. "Let's do more experimenting." was Mrs. Barnes answer to Mary's question.





Let's see what happens if we work backwards.



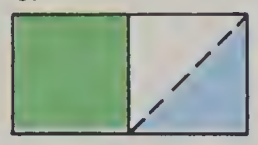


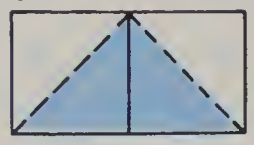
If we agree that an area this size suggests 1,

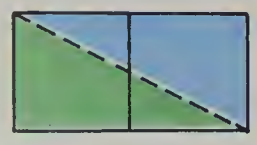
then what numbers are suggested by the following:

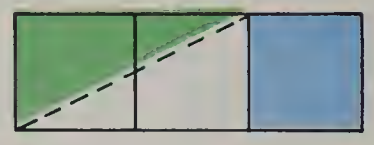
— area shaded green?
— area shaded blue?
— area not shaded?
— area shaded blue, plus area shaded green?
— area shaded blue, plus unshaded area?
— area shaded green, plus unshaded area?

a.

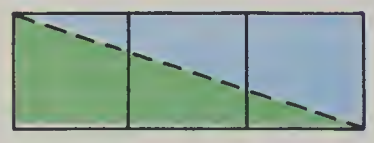
b.


c.


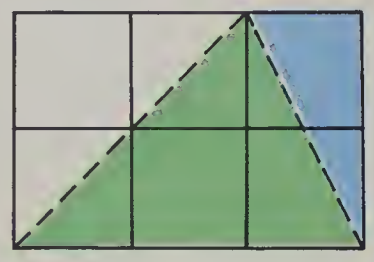
d.


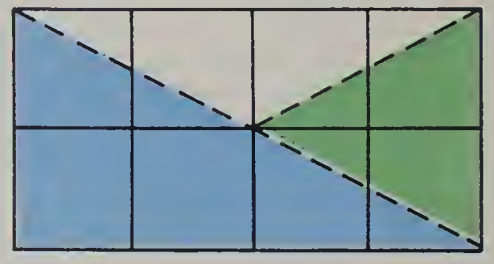
e.


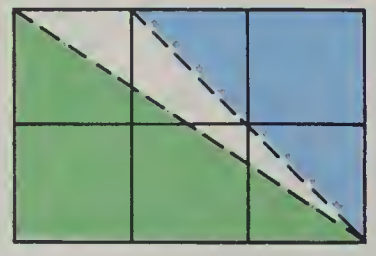
f.

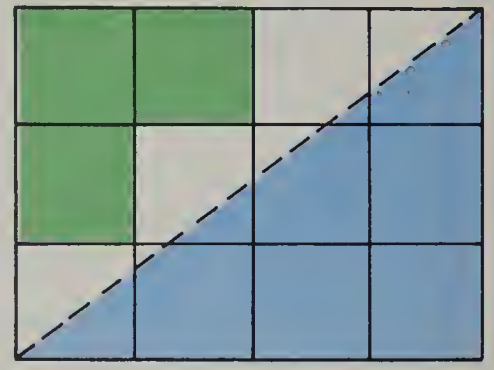
g.


h.

k.


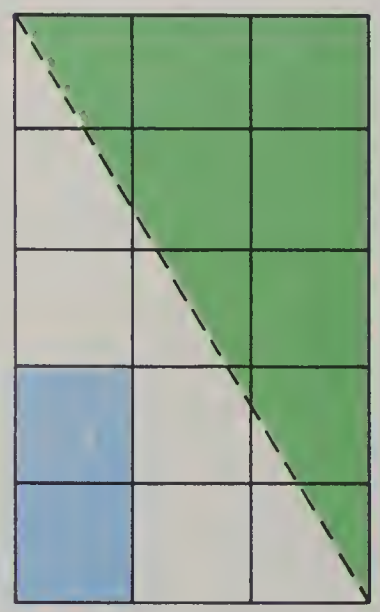
l.


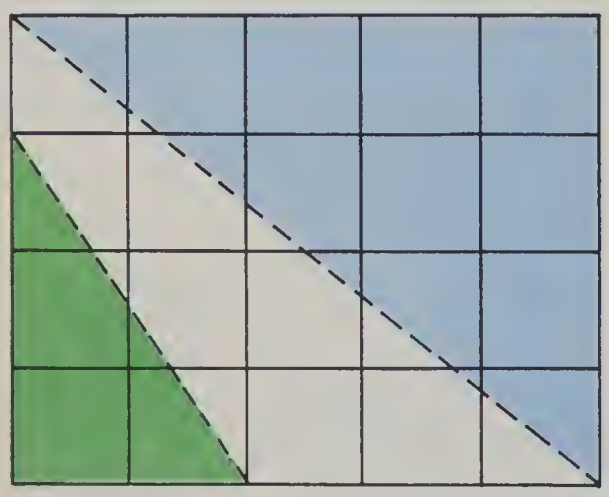
m.


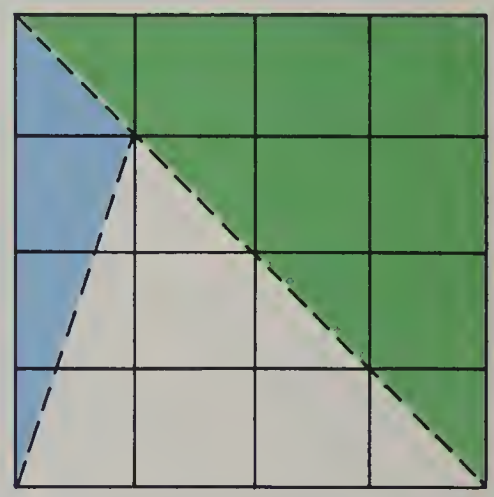
n.


i

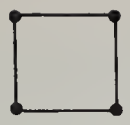
j

q.


o.


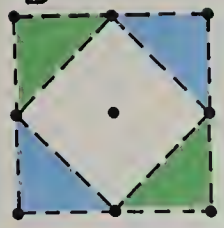
p.


	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q
area shaded in green suggests																	
area shaded in blue suggests																	
unshaded area suggests																	
total area suggests																	

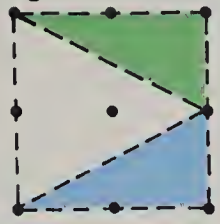
On this page we agree that  suggests 1.

A

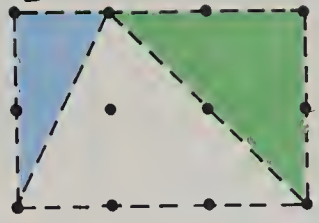
B



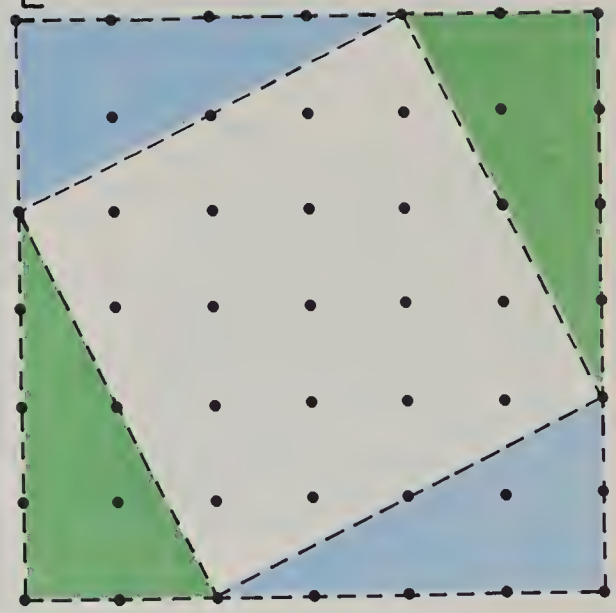
C



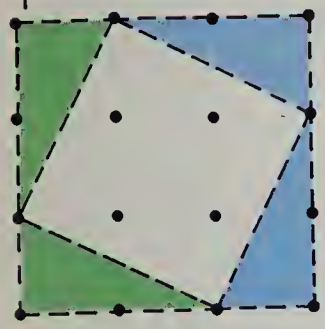
D



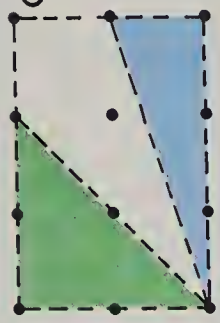
E



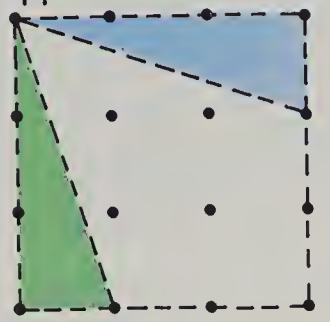
F



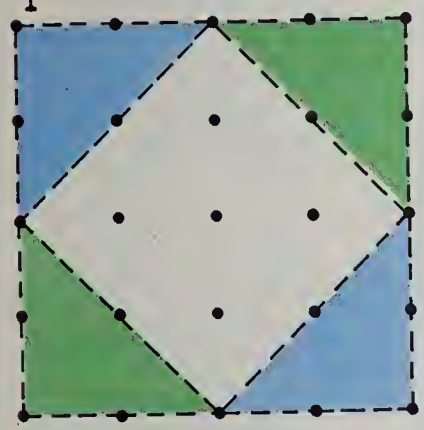
G



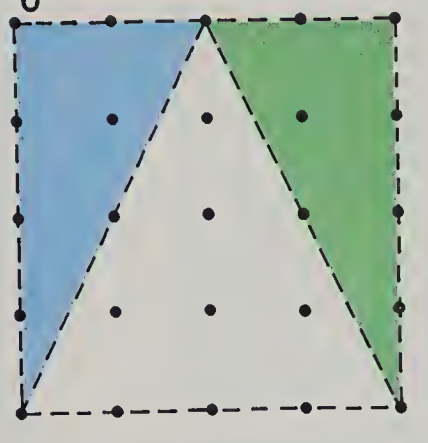
H



I



J



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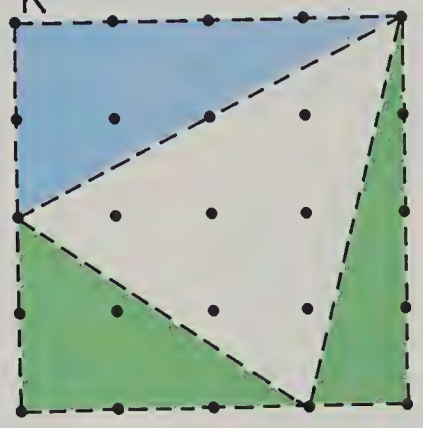

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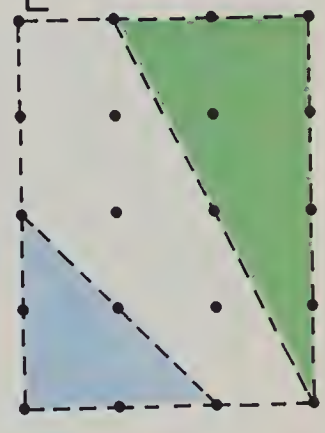
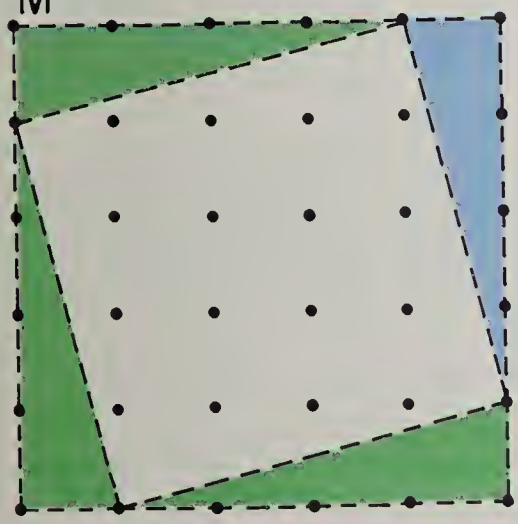


M



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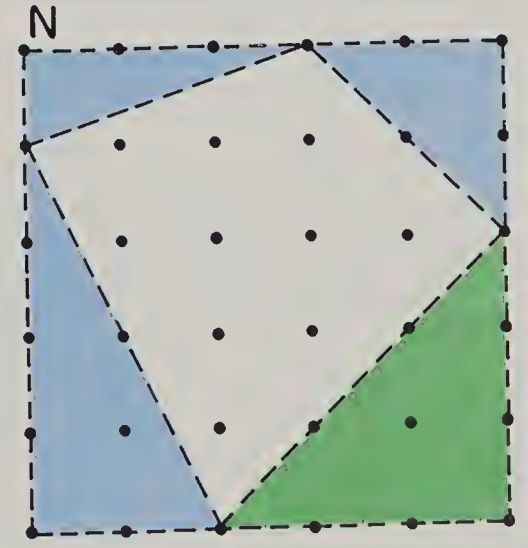
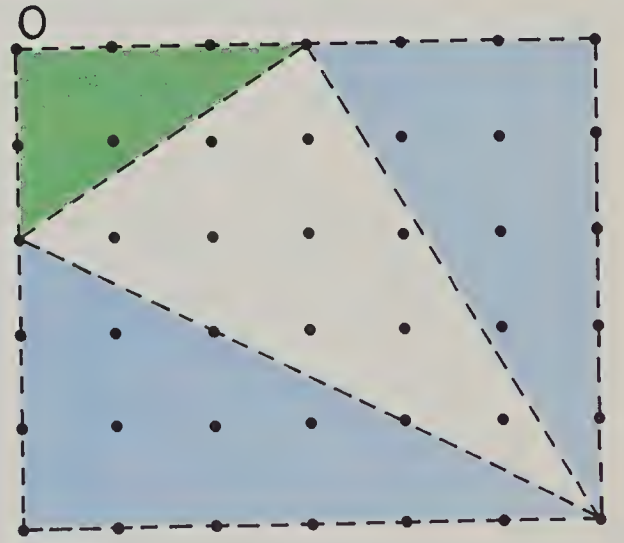
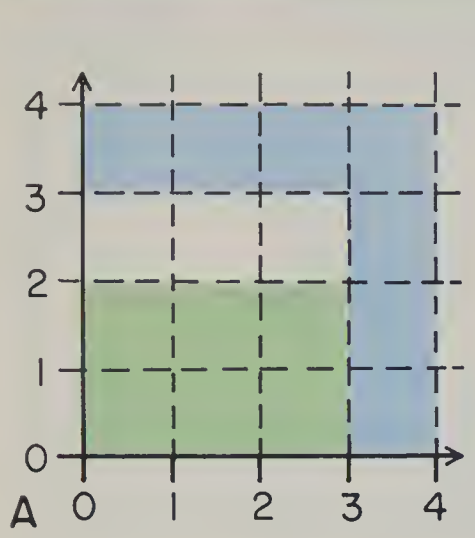


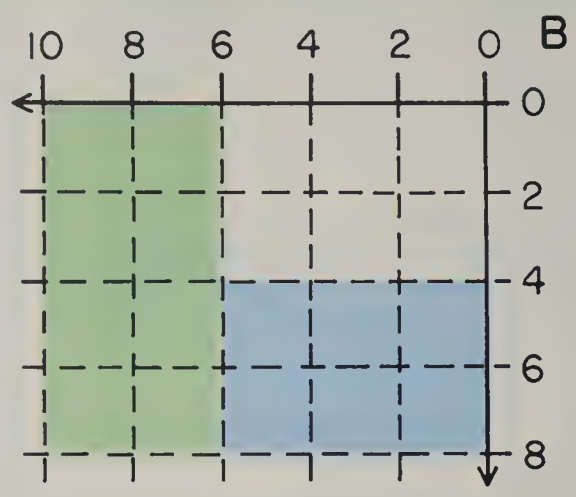
O

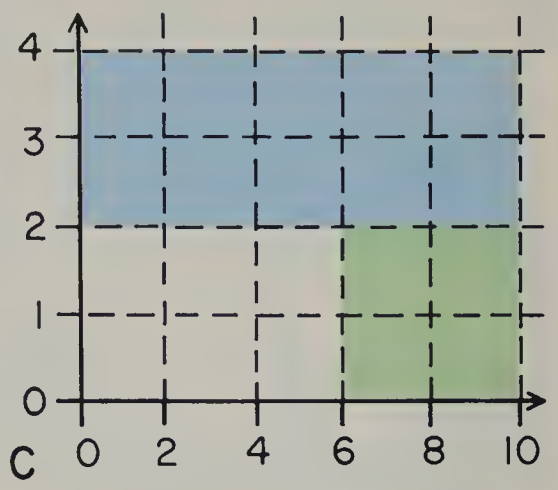


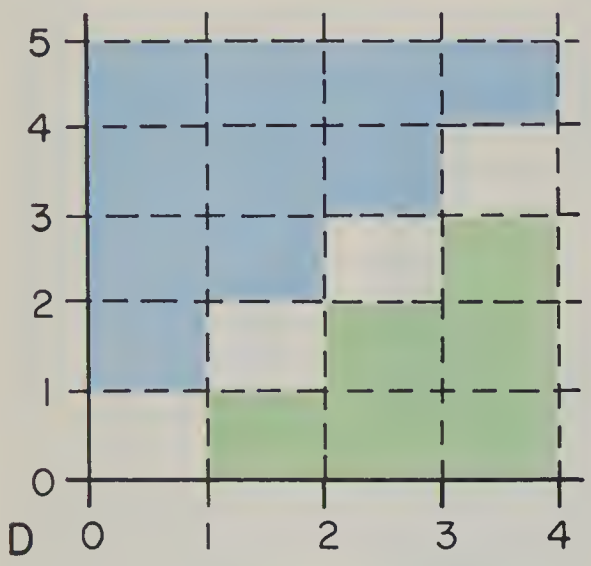
	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
Area shaded in green suggests															
Area shaded in blue suggests															
Unshaded area suggests															
Total area suggests															

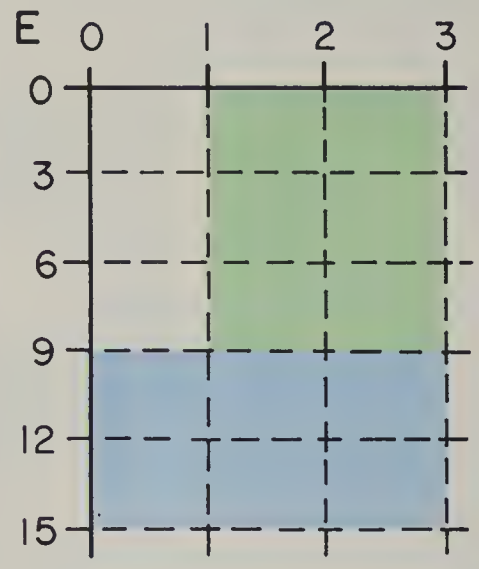
Addend		7	80	17			8		10	
Sum	100	100			100	100	48	72	200	200
Addend	10		10	60	80	15		12		11

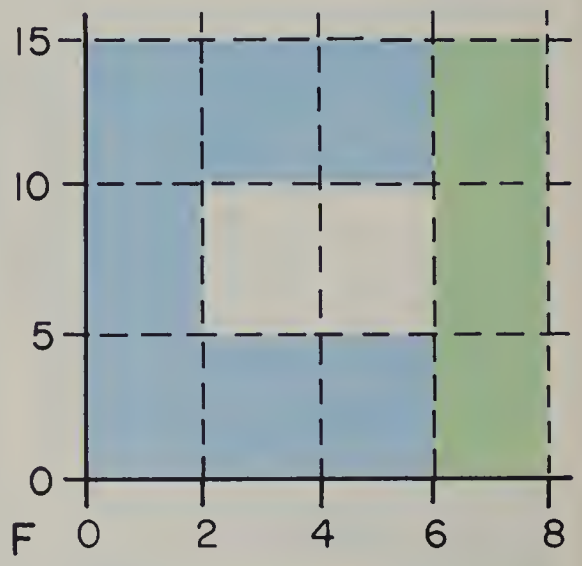










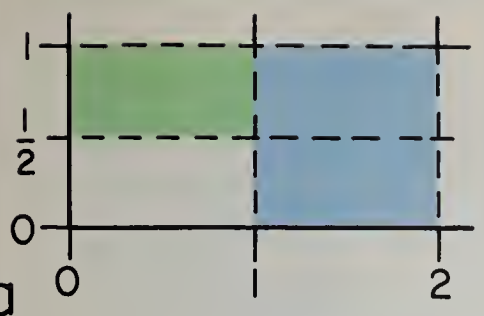


In each picture, the green, blue and unshaded areas suggest:

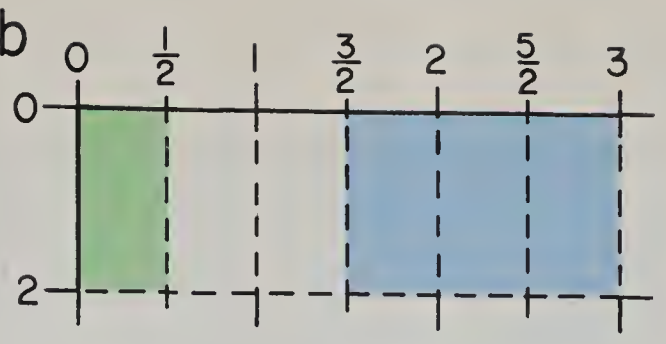
	□	□	□	□ + □	□ + □	□ + □	□ + □ + □
A							
B							
C							
D							
E							
F							

Sum			25	33			40	40			42	
Addend	2	16	10		28	4		17	6	24	5	95
Addend	18	5		29	2	17	7		16	11		7

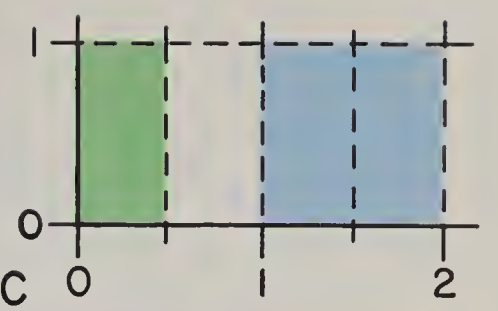
a



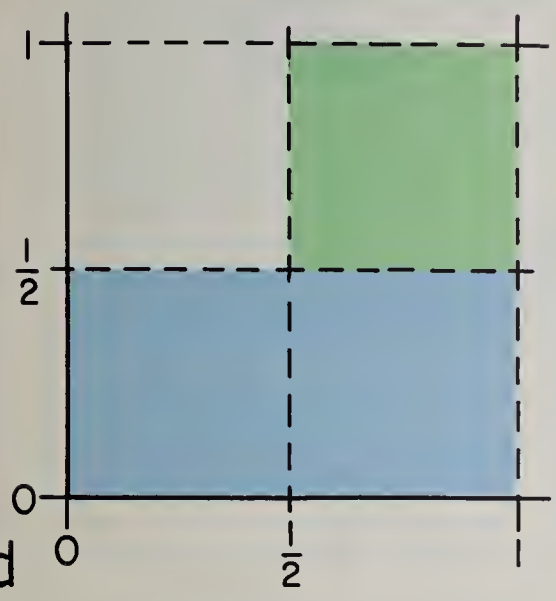
b



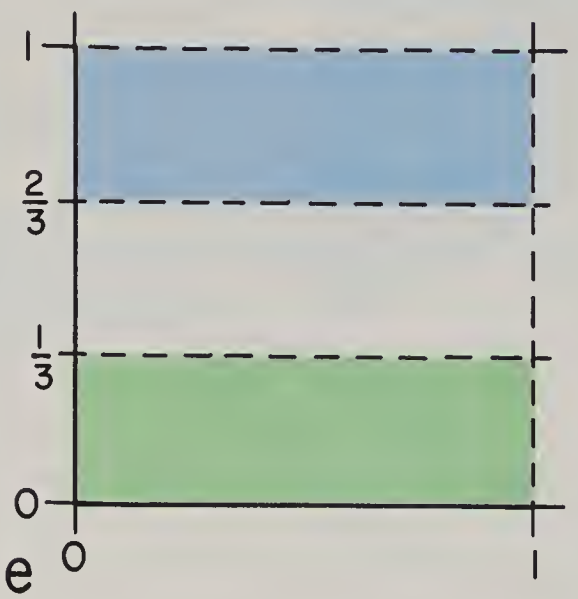
c



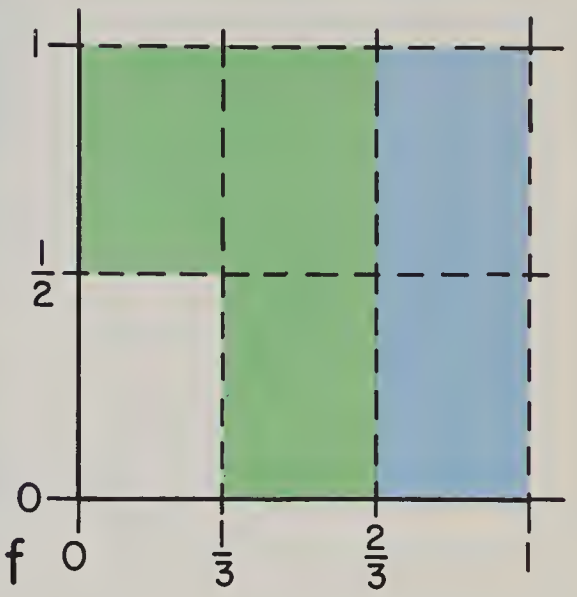
d



e



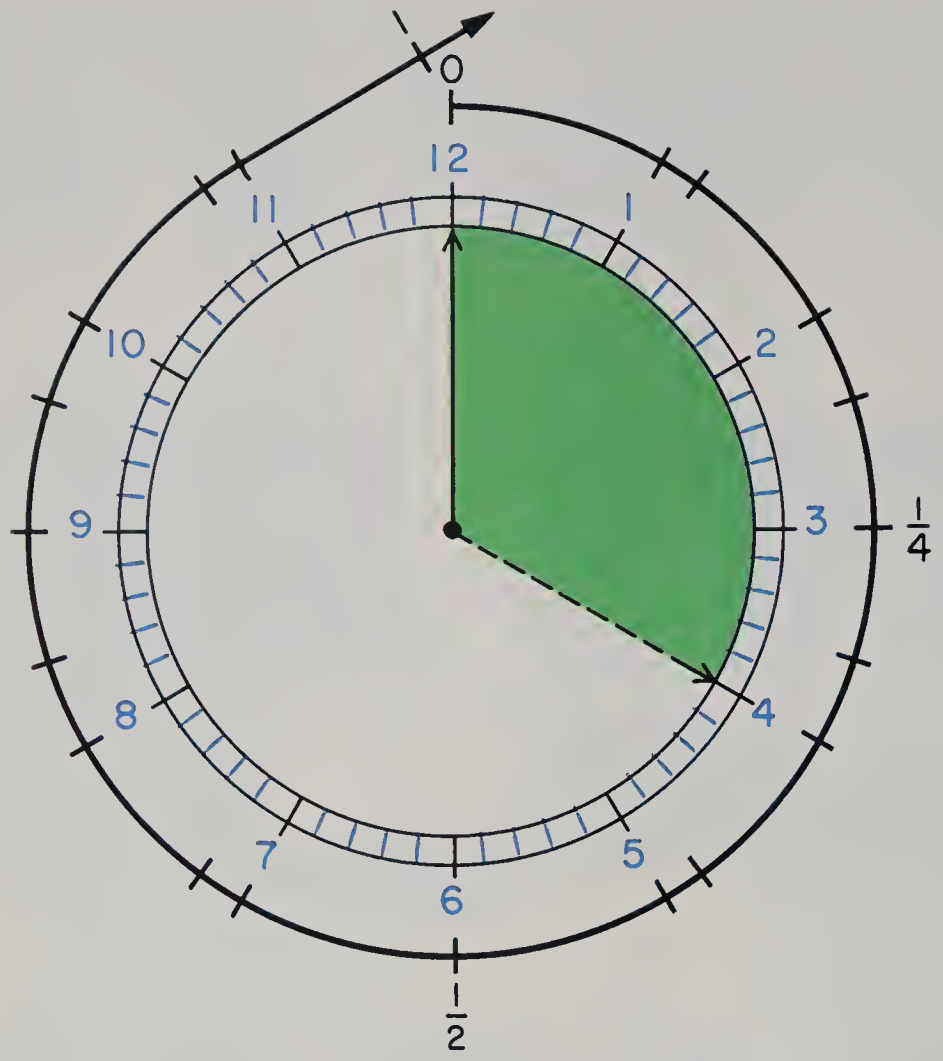
f



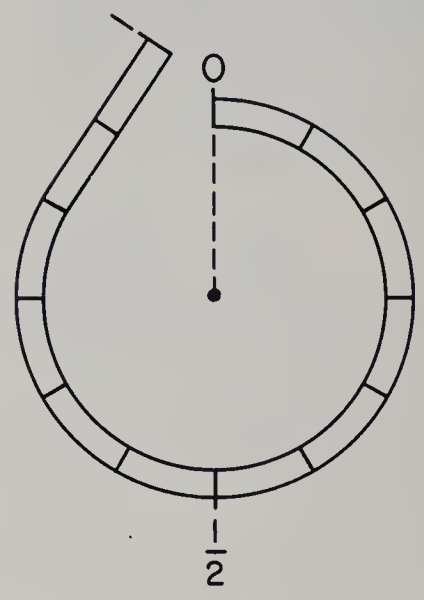
a						
b						
c						
d						
e						
f						

x						
	48		56			
		18	63	45		
			21			27
				20	32	

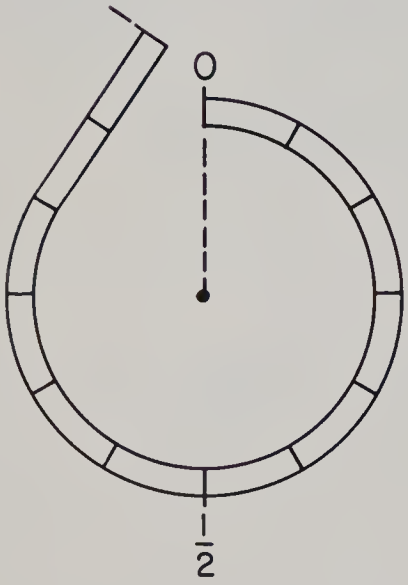
+	10	18	14	17	12
	18				
		24			
			21		
				26	



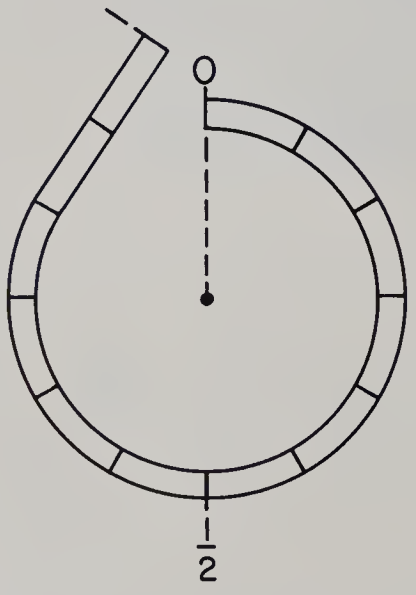
a. $\frac{1}{4} + \frac{1}{2} = \_\_$



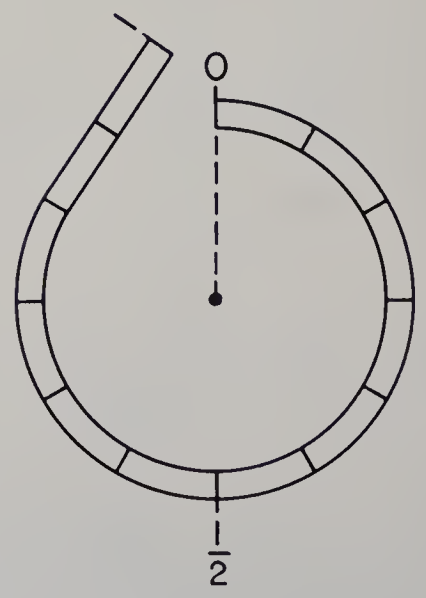
b. $\frac{1}{3} + \frac{1}{4} = \_\_$



d. $\frac{1}{12} + \frac{1}{3} = \_\_$



e. $\frac{5}{12} + \frac{1}{6} = \_\_$



c. $\frac{2}{5} + \frac{1}{10} = \_\_$

f. $\frac{1}{2} + \_\_ = \frac{3}{4}$

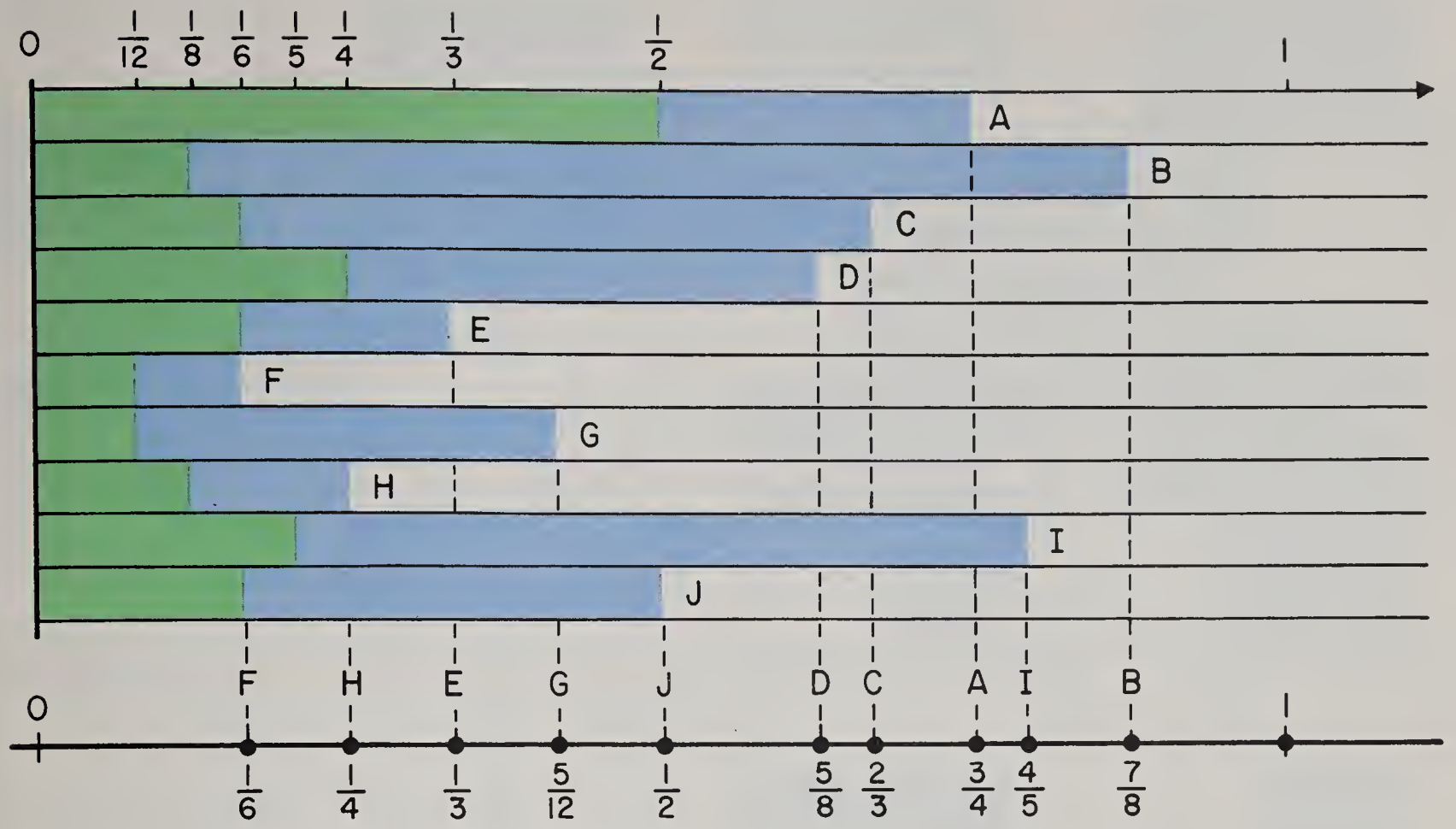
g. $\frac{1}{6} + \_\_ = \frac{1}{2}$

h. $\_\_ + \frac{1}{12} = \frac{3}{4}$

i. $\frac{2}{3} + \_\_ = \frac{11}{12}$

j. $\frac{1}{12} + \_\_ = \frac{1}{3}$

k. $\frac{1}{6} + \_\_ = \frac{2}{3}$



A. $\frac{1}{2} + \frac{1}{4} = -$ B. $- + - = -$ C. $- + - = -$ D. $- + - = -$

E. $- + - = -$ F. $- + - = -$ G. $- + - = -$ H. $- + - = -$

I. $- + - = -$ J. $- + - = -$ K. $- + - = -$ L. $- + - = -$

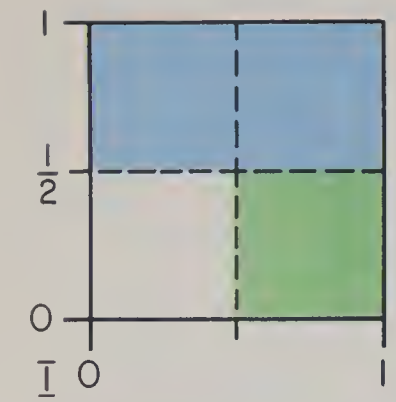
Another way of looking at the same pictures:

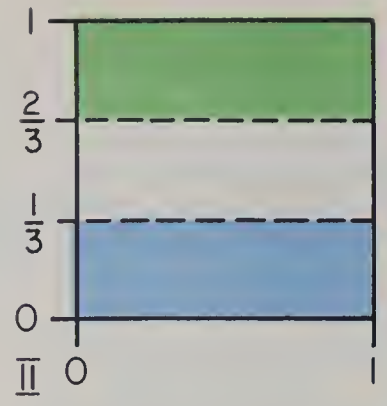
A. $- - - = -$ B. $- = - - -$ C. $- - - = -$ D. $- = - - -$

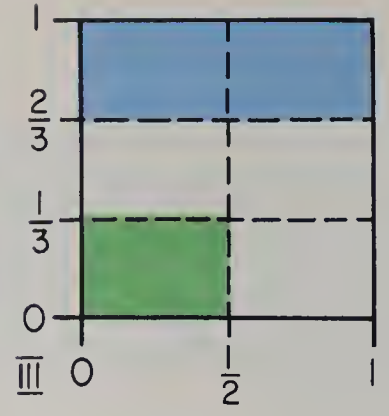
E. $- - - = -$ F. $- = - - -$ G. $- = - - -$ H. $- - - = -$

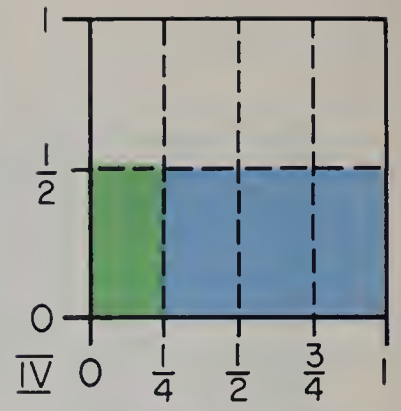
I. $- - - = -$ J. $- = - - -$ K. $- - - = -$ L. $- = - - -$

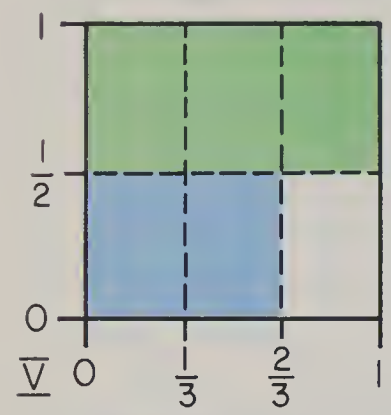
			10	16	17	15	20	25	29	52	38	Sum
Factor						8						Addend
Factor	2											Addend
Product	24	13	21	63	0		100	100	100	100	72	

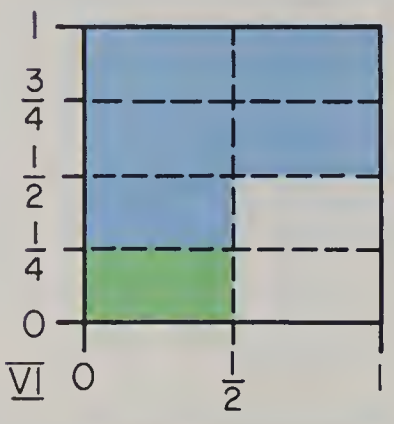


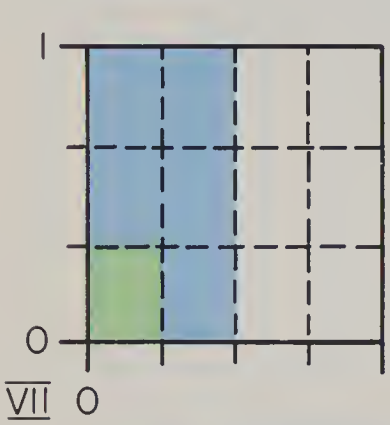


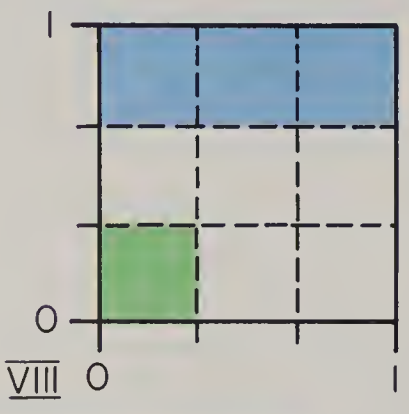


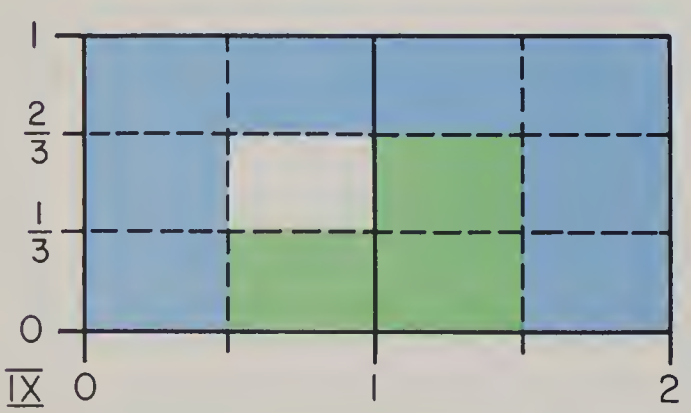


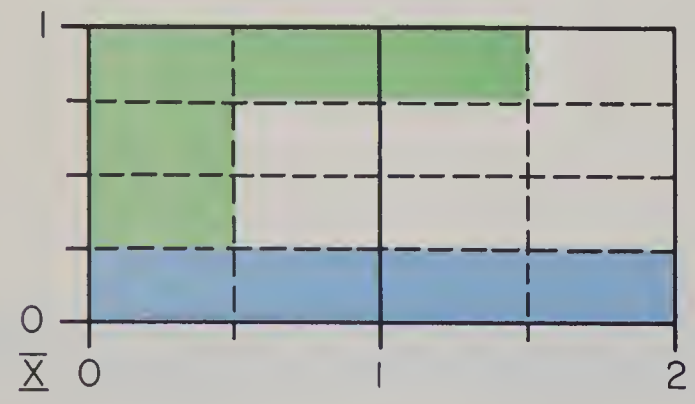


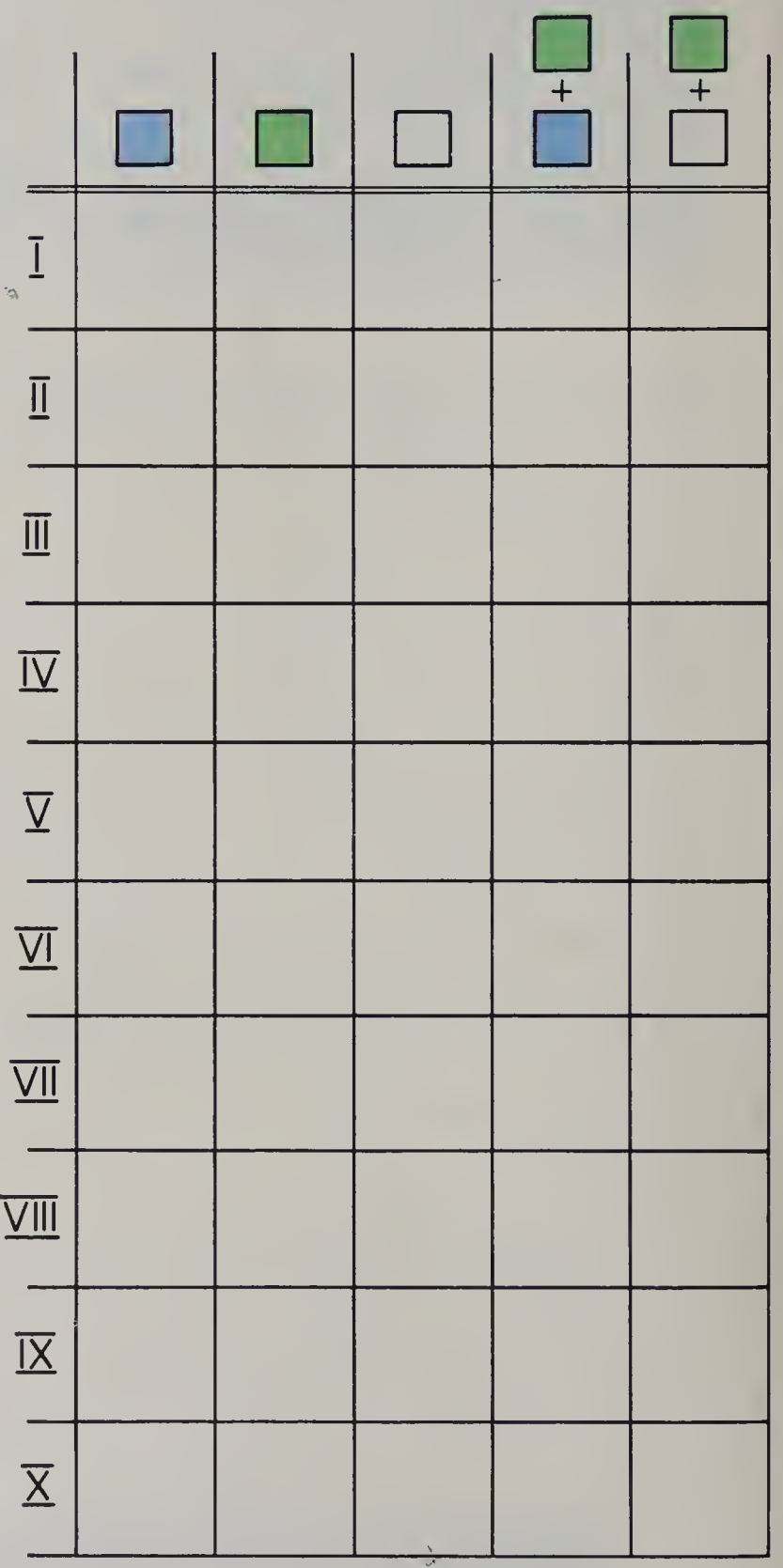






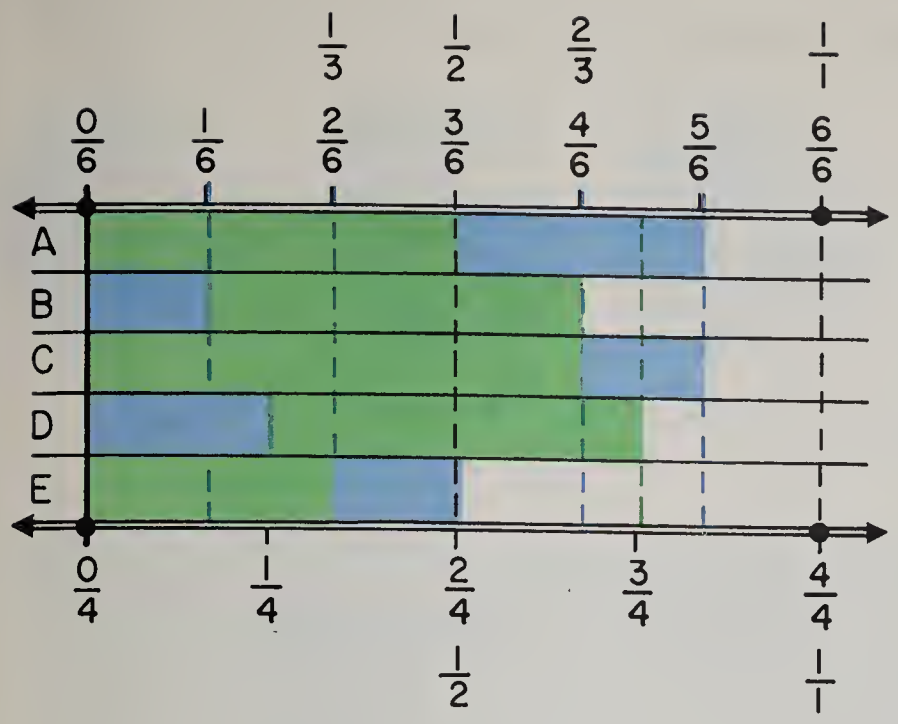


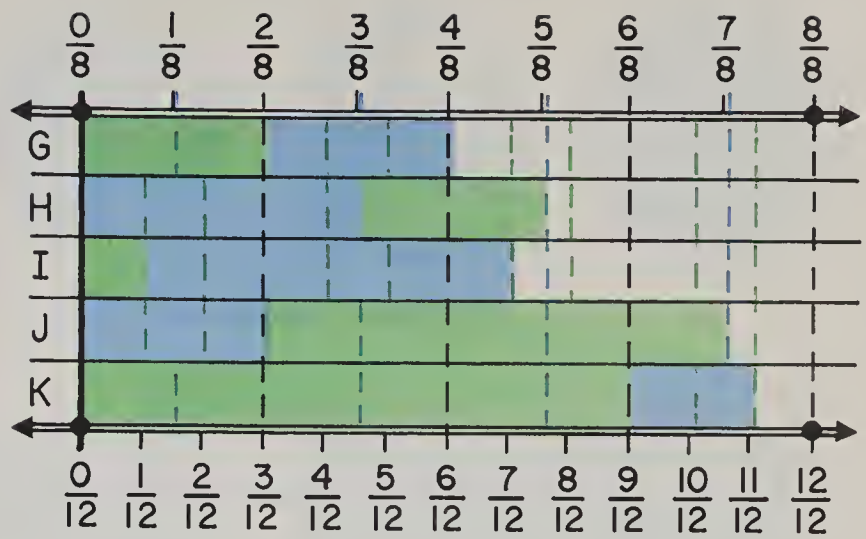




All factors are greater than two!

Product	21	49	27	42	63	81	35	39	45	56
Factor										
Factor										





A. $\frac{1}{2} + \frac{1}{3} = —$

B. $\frac{1}{6} + \frac{1}{3} = —$

G. $\frac{2}{8} + \frac{3}{12} = —$

H. $\frac{3}{8} + \frac{1}{4} = —$

C. $\frac{2}{3} + \frac{1}{6} = —$

D. $\frac{1}{4} + — = \frac{3}{4}$

I. $\frac{1}{12} + \frac{1}{2} = —$

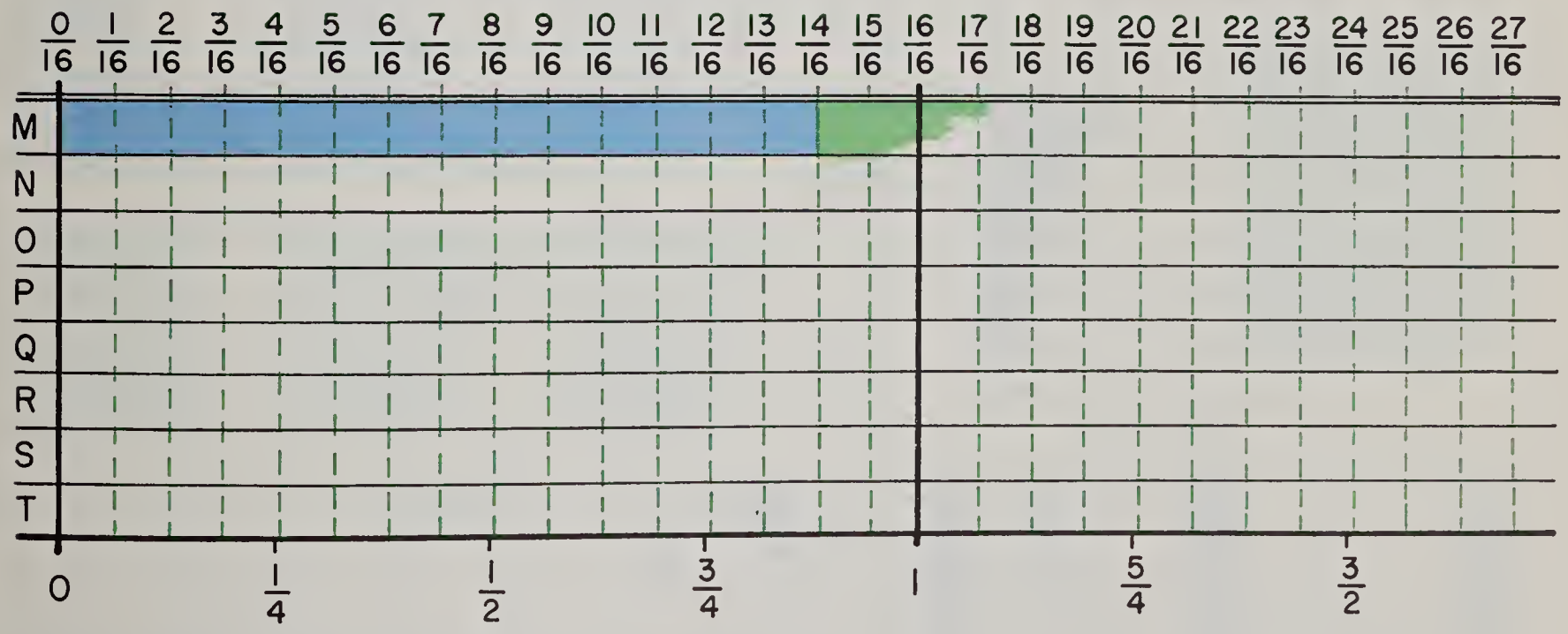
J. $\frac{1}{4} + — = \frac{7}{8}$

E. $\frac{1}{3} + — = \frac{1}{2}$

F. $\frac{2}{6} + \frac{2}{3} = —$

K. $\frac{3}{4} + \frac{1}{6} = —$

L. $— + \frac{5}{12} = \frac{4}{4}$



M. $\frac{7}{8} + \frac{1}{4} = —$

N. $\frac{13}{16} + \frac{1}{2} = —$

O. $\frac{9}{16} + \frac{9}{16} = —$

P. $\frac{5}{16} + \frac{11}{16} = —$

Q. $\frac{3}{4} + — = \frac{15}{16}$

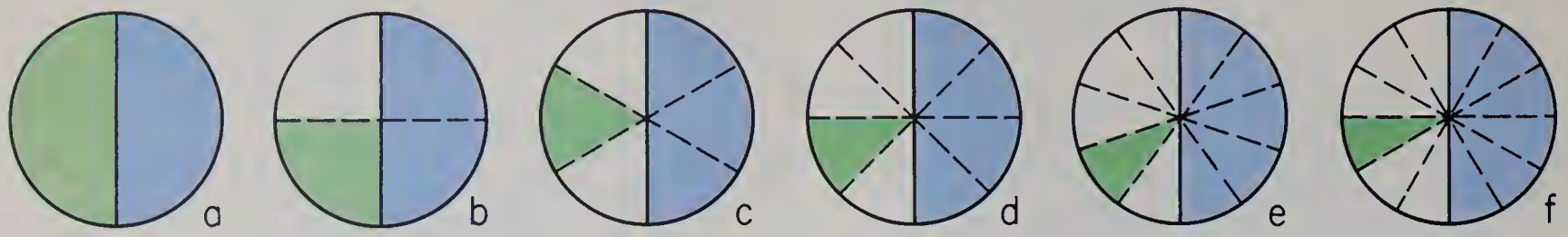
R. $— + \frac{1}{8} = \frac{3}{4}$

S. $\frac{4}{16} + — = \frac{3}{8}$

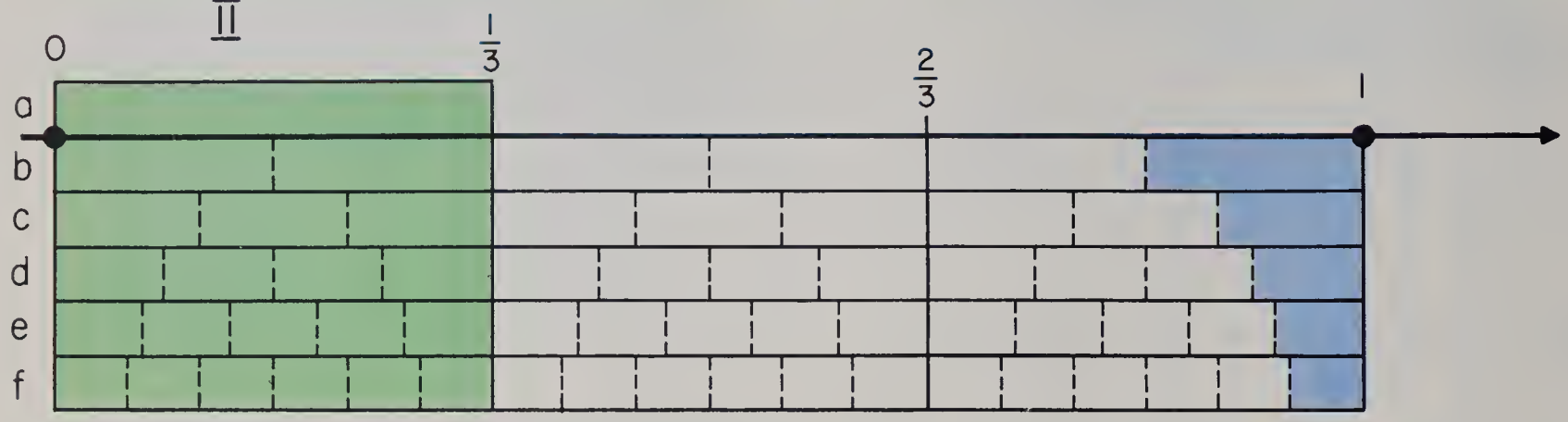
T. $\frac{5}{8} + — = \frac{3}{2}$

If fractions are numbers, they must have many names!

I

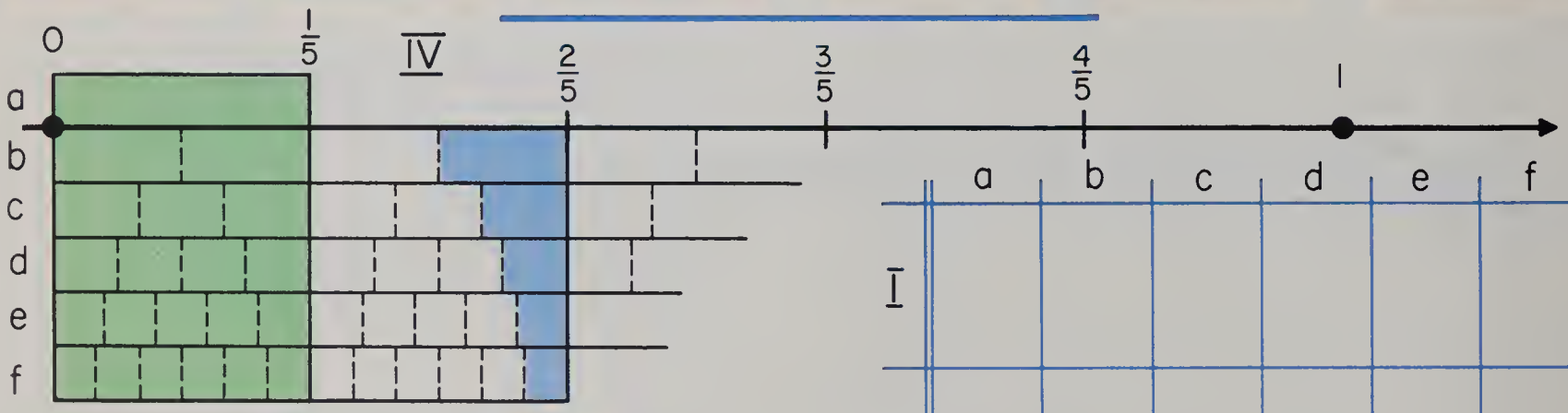


II

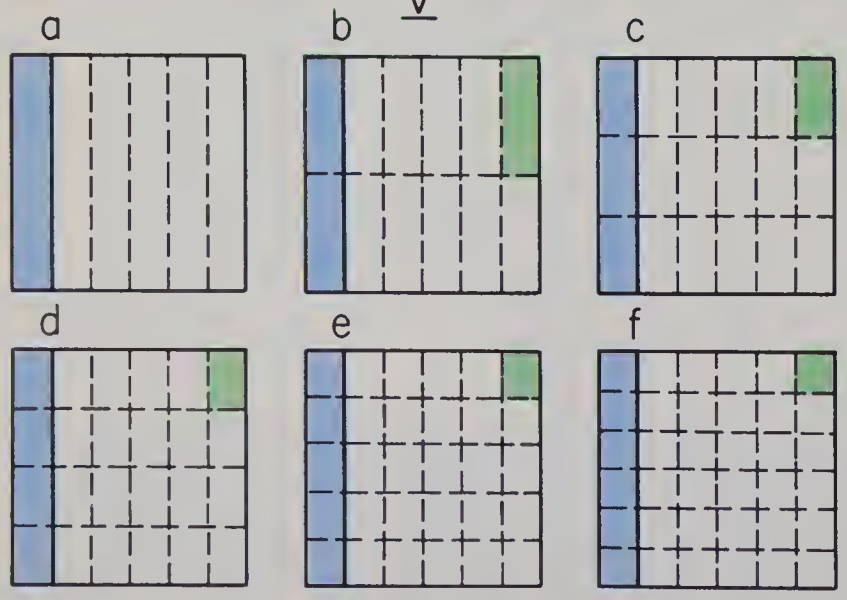


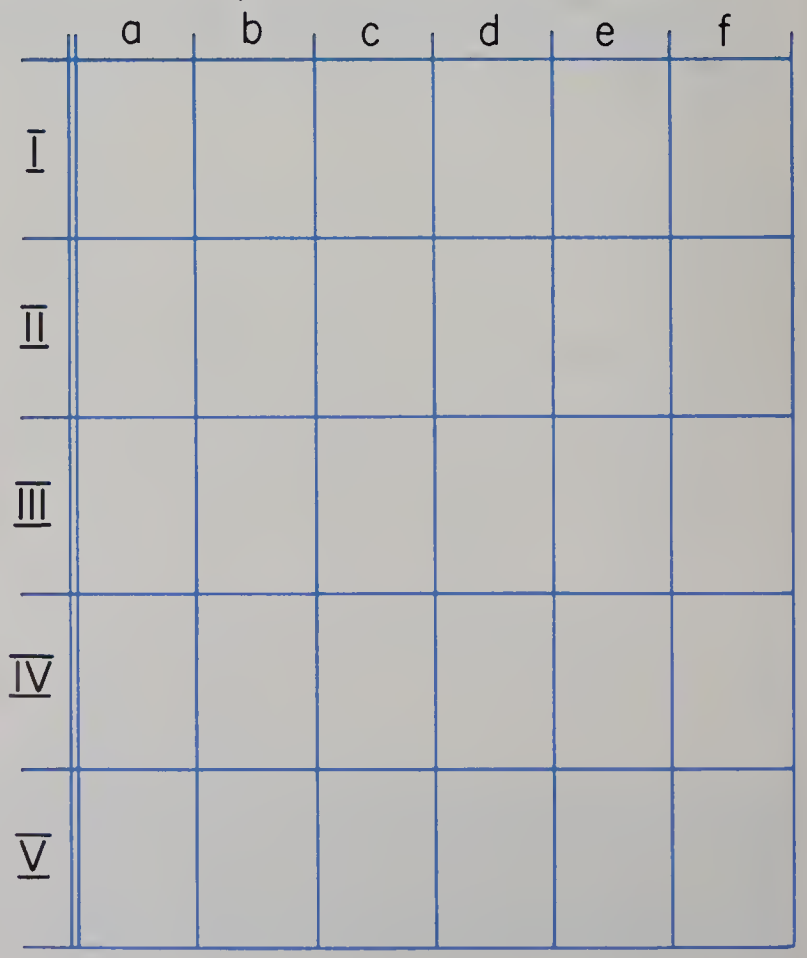
III

IV

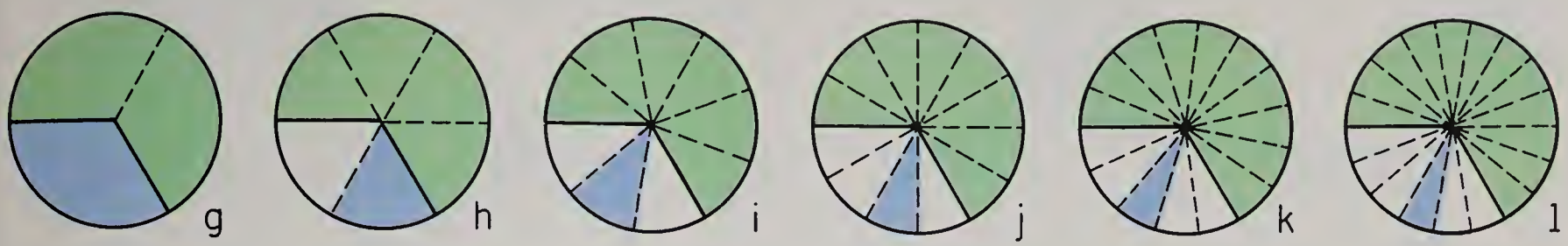


V

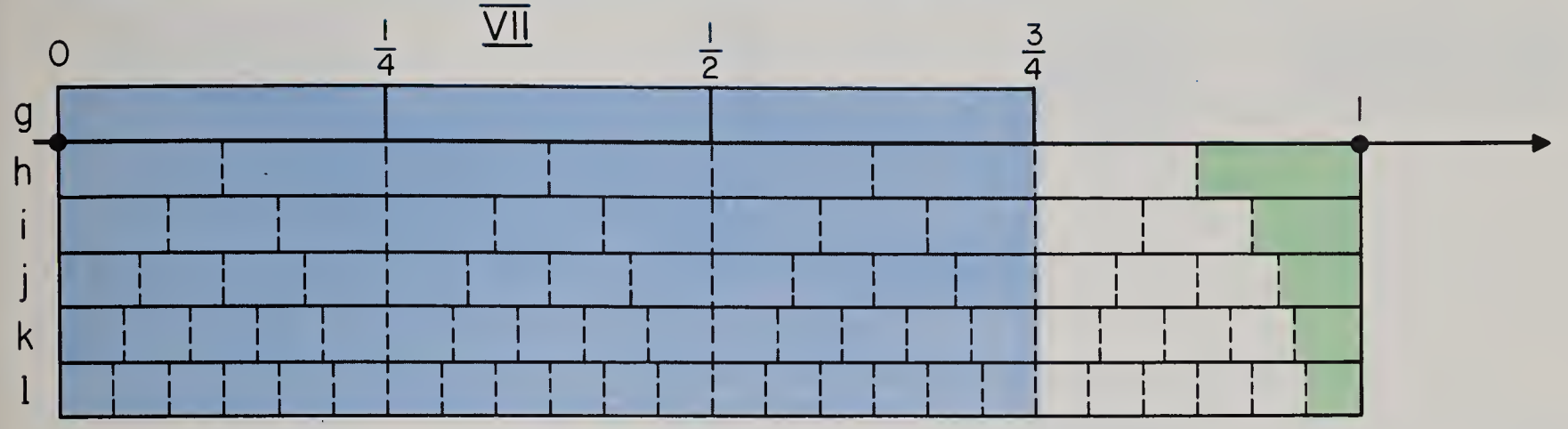




VI



VII



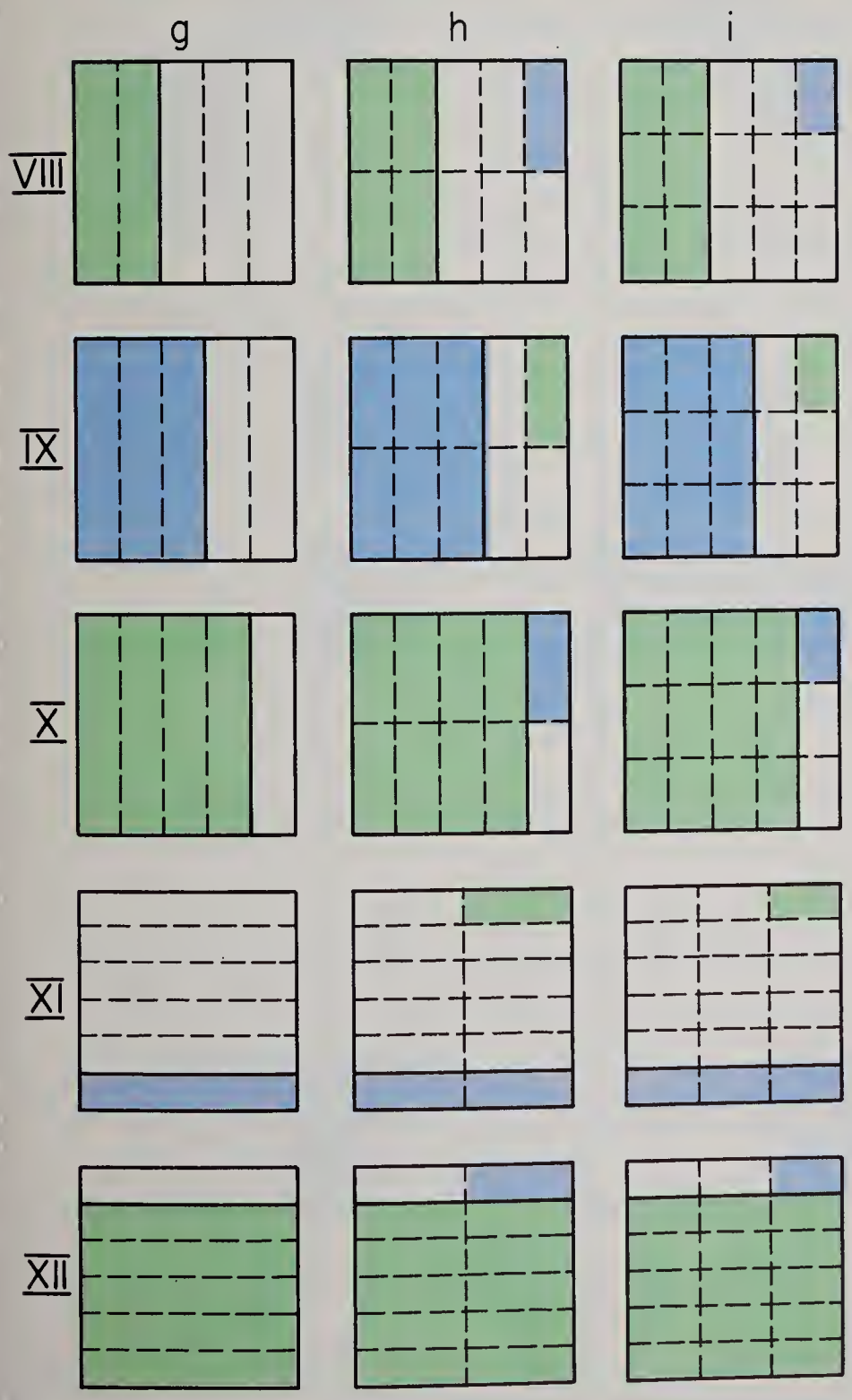
VIII

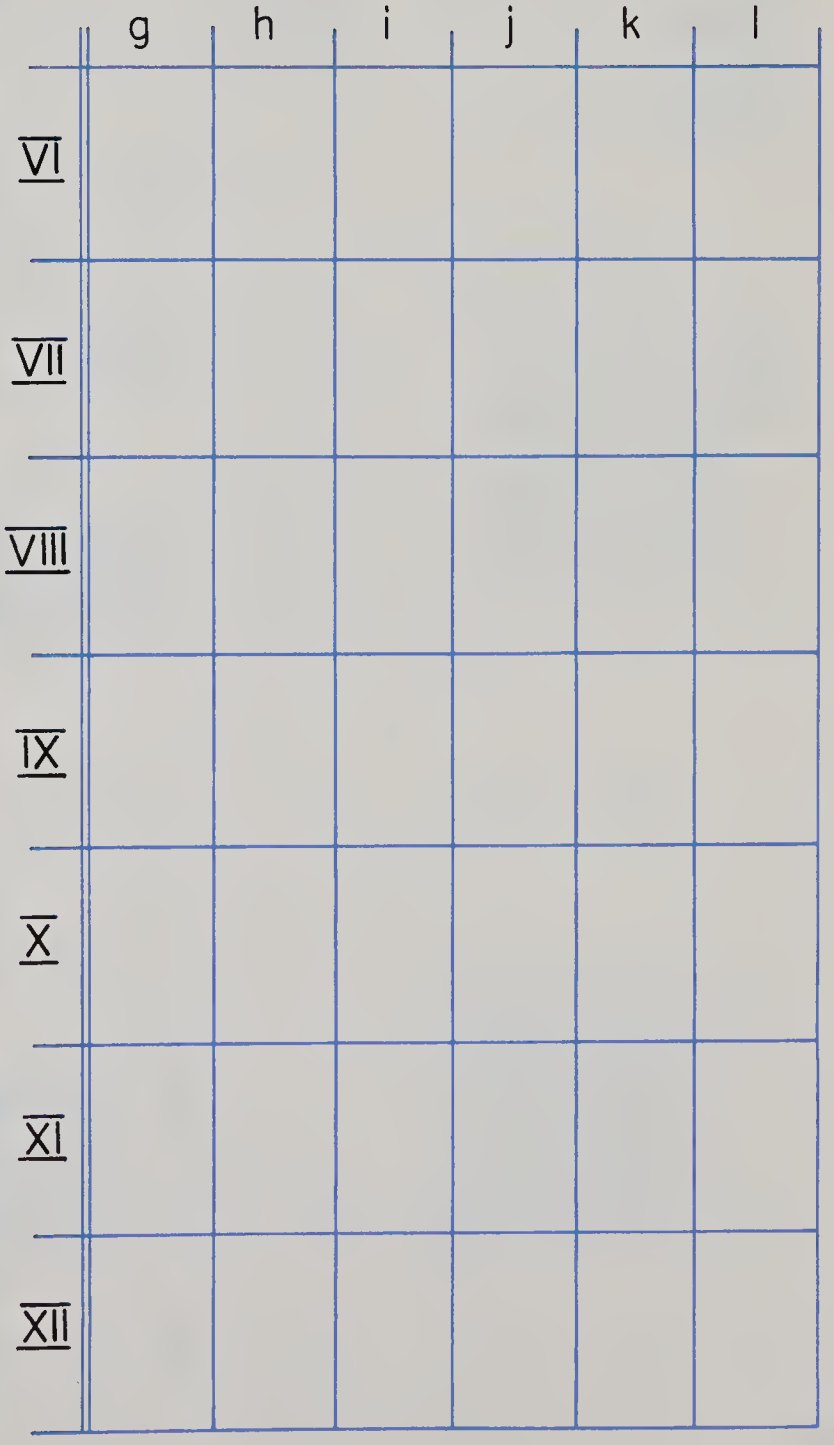
IX

X

XI

XII



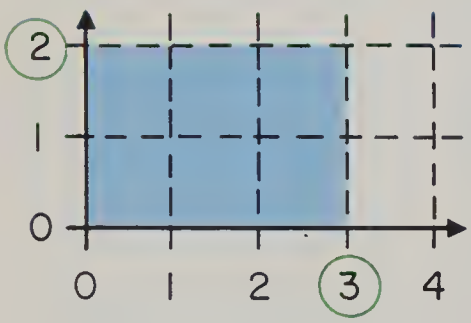


If fractions are numbers, how could "multiplication of fractions" be interpreted?

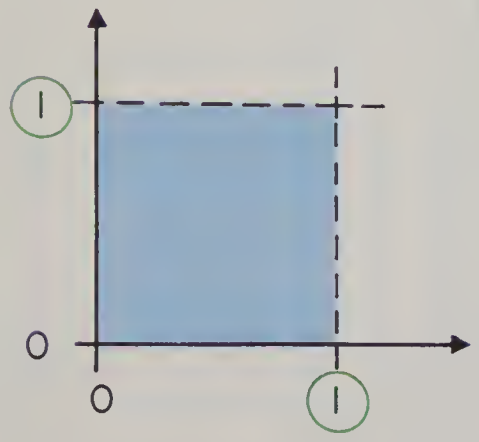
Here are some questions that suggest multiplication:		Here are possible interpretations:
I. How much is $\frac{1}{2}$ of a half-dollar?	____	$\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$
II. In two groups of 10, there are 20. How many are there in $\frac{1}{2}$ of a group of 10?	____	$\frac{1}{2} \times 10 = 5$
III. How many inches are there in $\frac{1}{3}$ of a foot?	____	$\frac{1}{3} \times 12 = 4$
IV. How many days in $\frac{2}{7}$ of a week?	____	$\frac{2}{7} \times 7 = 2$
V. "How many quarters have you?" Bill asked. "Five of them," Mary said. How much does she have?	____	$5 \times \frac{1}{4} = \frac{5}{4}$
VI. Jack said, "I've read $\frac{1}{3}$ of the book." Gerry said he had read $\frac{1}{2}$ as far. "I've read twice as far as Jack," Alice said. How far has Gerry read? How far has Alice read?	____	$\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$
	____	$2 \times \frac{1}{3} = \frac{2}{3}$

Pictures that suggest the idea:

Multiplication of whole numbers

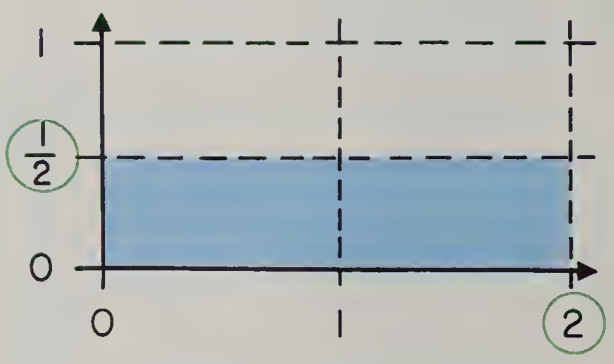


$3 \times 2 = 6$

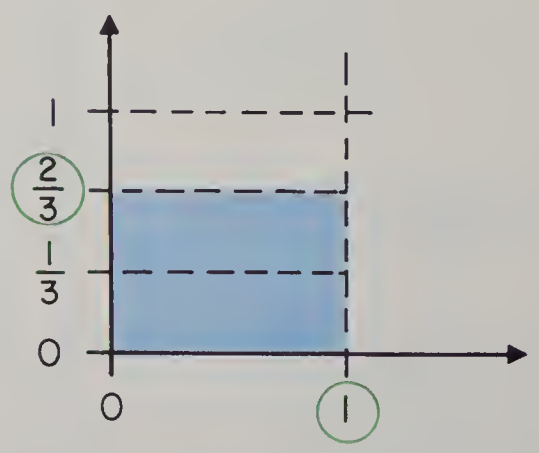


$1 \times 1 = 1$

Multiplication of a whole number and a fraction

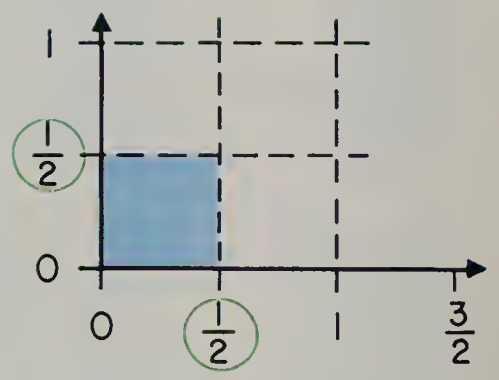


$2 \times \frac{1}{2} = \frac{2}{2} = 1$

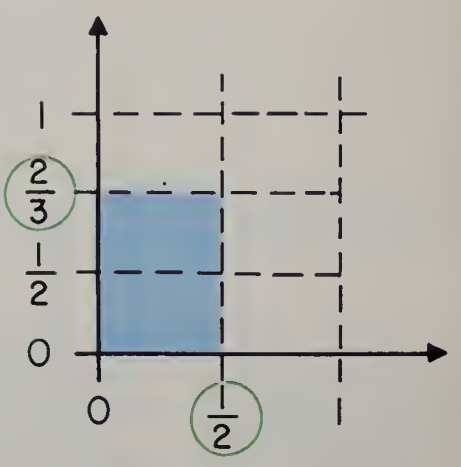


$1 \times \frac{2}{3} = \frac{2}{3}$

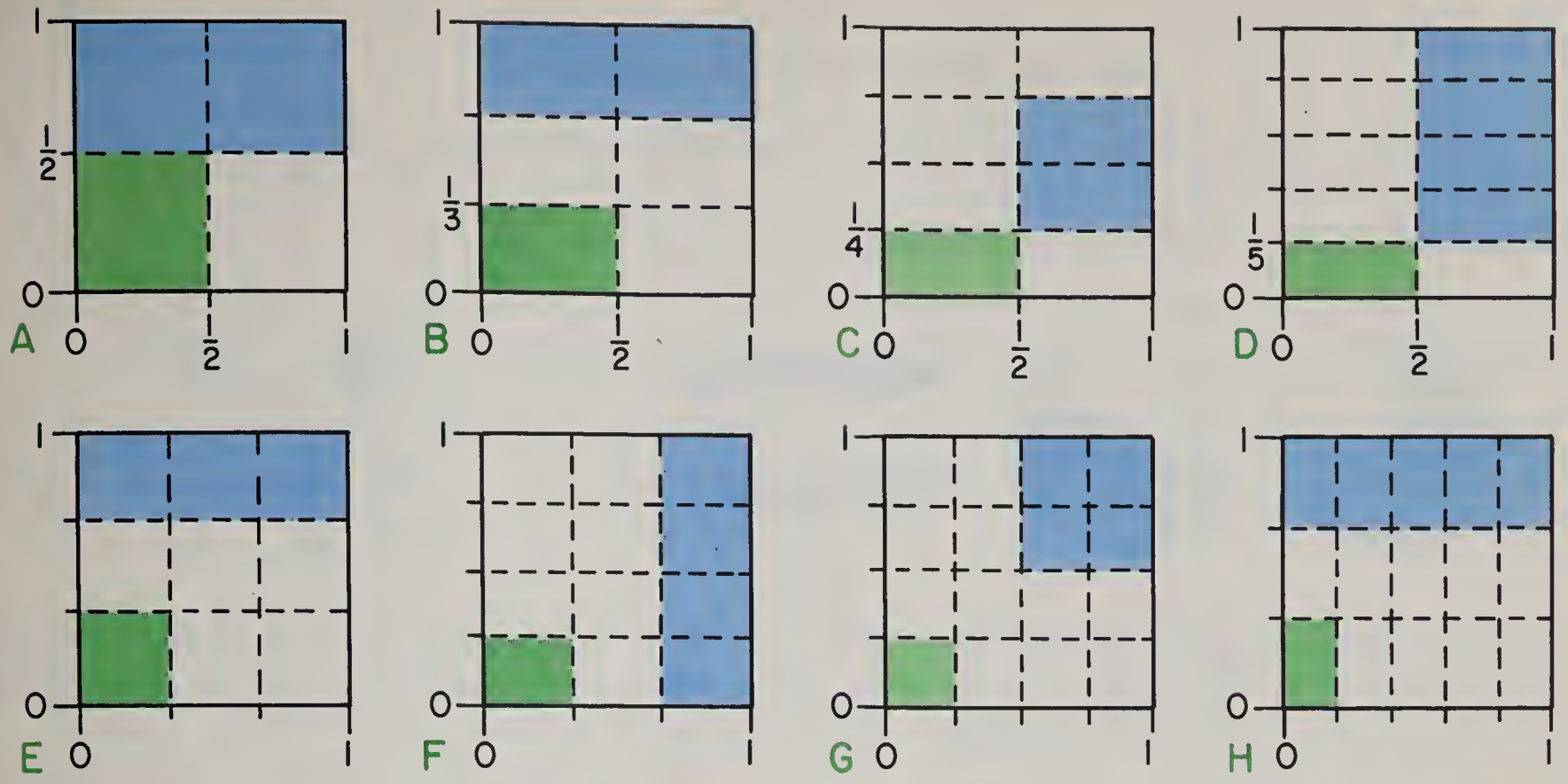
Multiplication of two fractions



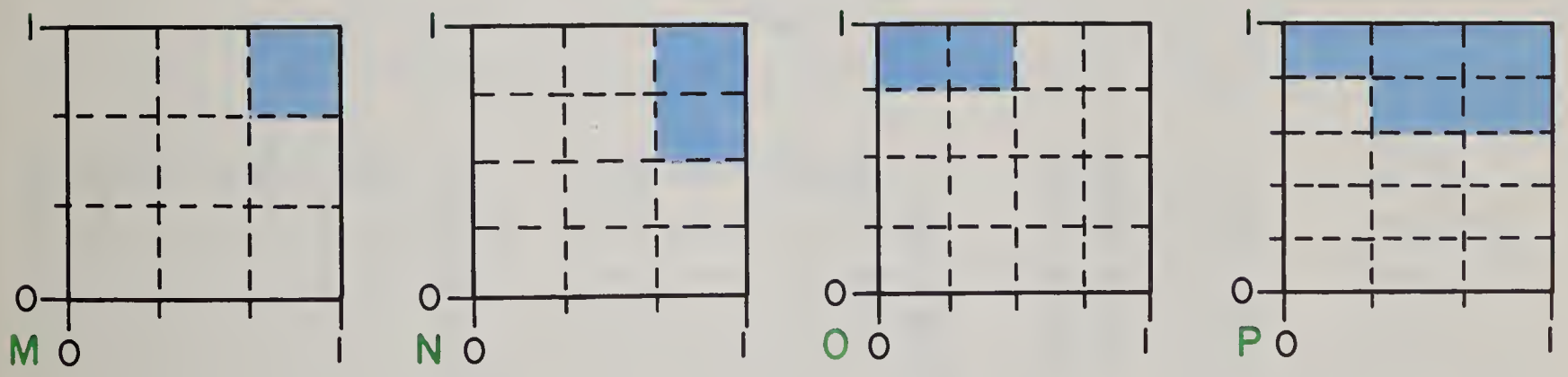
$\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$



$\frac{1}{2} \times \frac{2}{3} = \frac{2}{6}$



Color area green suggested by mathematical statement below:



A $\frac{1}{2} \times \frac{1}{2} = \frac{\quad}{\quad}$ B $\frac{1}{2} \times \frac{1}{3} = \frac{\quad}{\quad}$ C $\frac{1}{2} \times \frac{1}{4} = \frac{\quad}{\quad}$ D $\frac{1}{2} \times \frac{1}{5} = \frac{\quad}{\quad}$

E $\frac{1}{3} \times \frac{1}{3} = \frac{\quad}{\quad}$ F $\frac{1}{3} \times \frac{1}{4} = \frac{\quad}{\quad}$ G $\frac{1}{4} \times \frac{1}{4} = \frac{\quad}{\quad}$ H $\frac{1}{5} \times \frac{1}{3} = \frac{\quad}{\quad}$

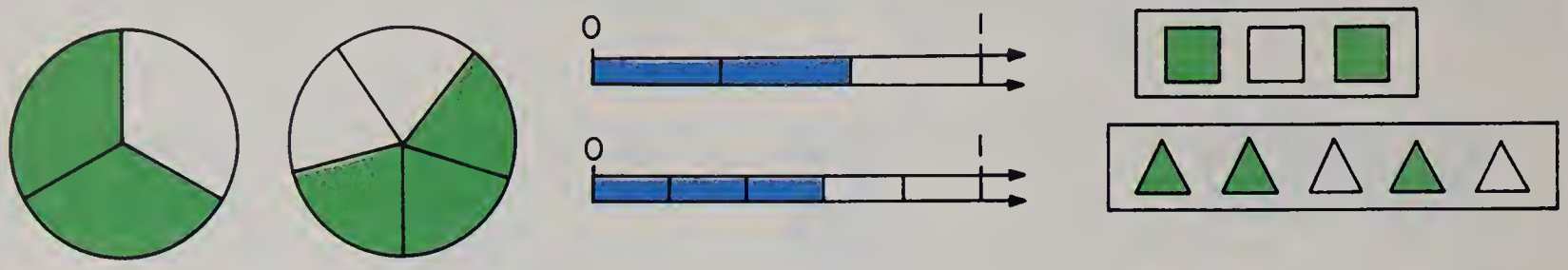
I $\frac{1}{2} \times \frac{2}{3} = \frac{\quad}{\quad}$ J $\frac{1}{2} \times \frac{3}{4} = \frac{\quad}{\quad}$ K $\frac{1}{3} \times \frac{2}{3} = \frac{\quad}{\quad}$ L $\frac{1}{4} \times \frac{3}{4} = \frac{\quad}{\quad}$

M $\frac{2}{3} \times \frac{2}{3} = \frac{\quad}{\quad}$ N $\frac{2}{3} \times \frac{3}{4} = \frac{\quad}{\quad}$ O $\frac{3}{4} \times \frac{3}{4} = \frac{\quad}{\quad}$ P $\frac{2}{3} \times \frac{2}{5} = \frac{\quad}{\quad}$

Make a list of the fractions suggested by the areas shaded in blue.

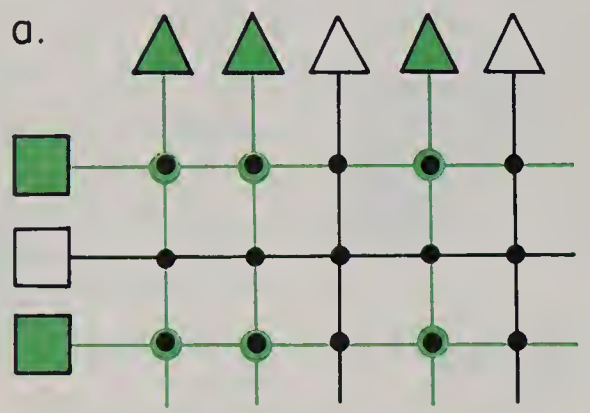
Greg makes a suggestion:

"There are many pictures that suggest fractions," Greg Wilson volunteered one day. "Each picture of two-thirds or three-fifths can be seen as 2 out of 3 or 3 out of 5. The following pictures are examples:"



"So, where I want to multiply two fractions such as two-thirds or three-fifths, I draw a picture that also suggests 2 out of 3 and 3 out of 5. I count the total number of crosses, first. Then I count those made by two colored lines.

a.



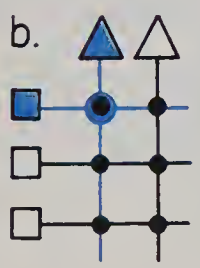
"There are 15 crossing points. There are six of these made by two colored lines drawn from 2 out of 3 and from 3 out of 5.

"So, I say, 6 out of 15 are made by the colored lines.

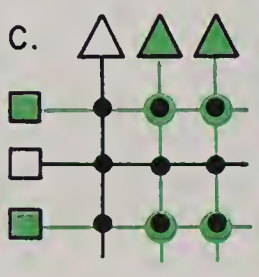
"And, I say that two-thirds times three-fifths must be six-fifteenths . . . or two-fifths."

"Here are some examples I tried," Greg said. "Each of these lead me to the same result I get by any other method."

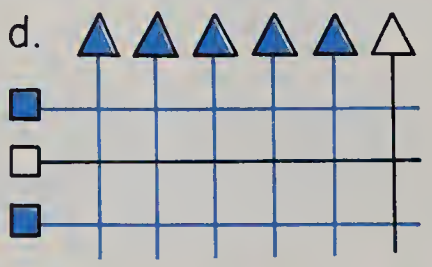
b.



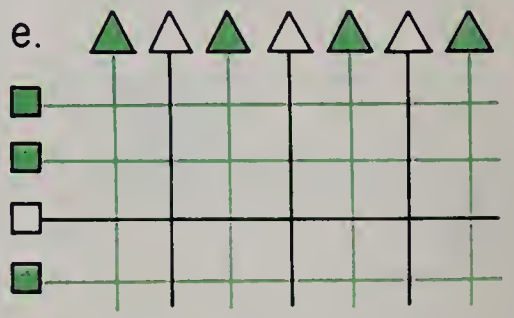
c.



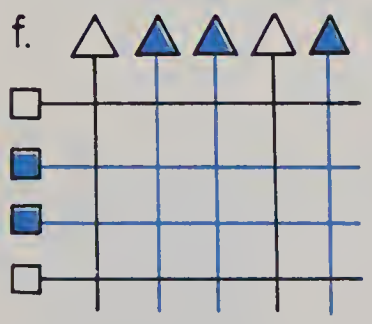
d.



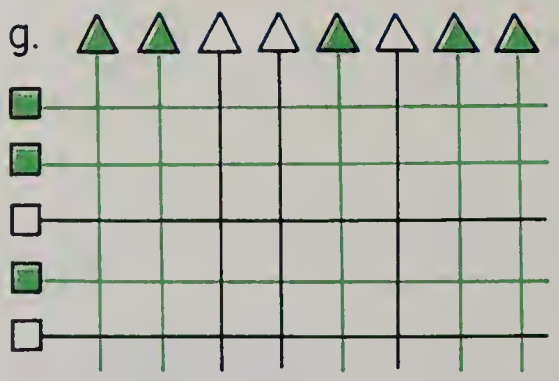
e.



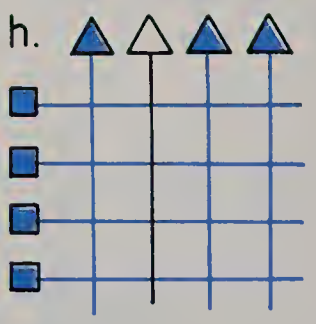
f.



g.



h.

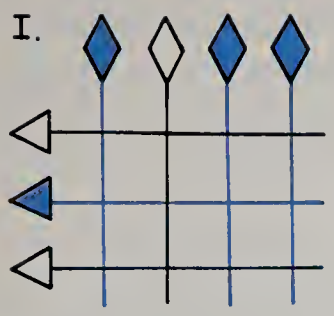


a. — x — = — b. — x — = — c. — x — = — d. — x — = —

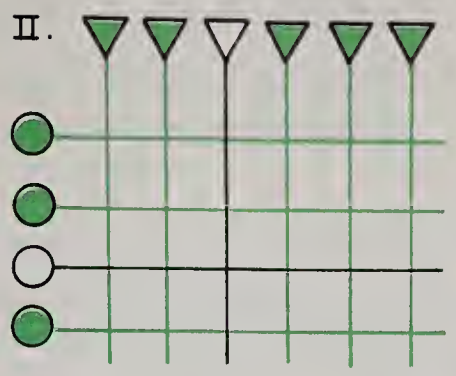
e. — x — = — f. — x — = — g. — x — = — h. — x — = —

Test Greg's pictures with more examples.

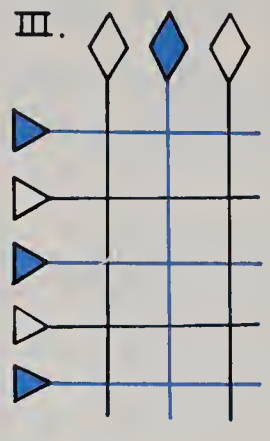
I.



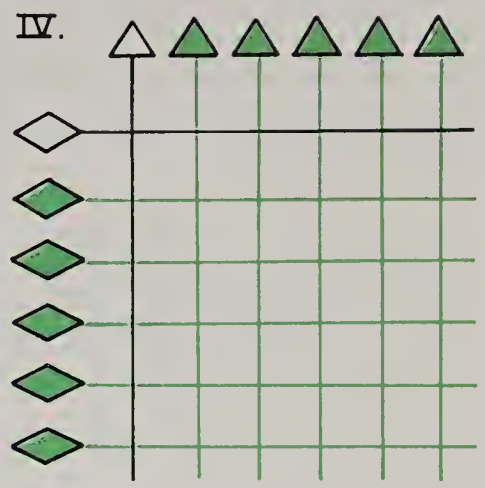
II.



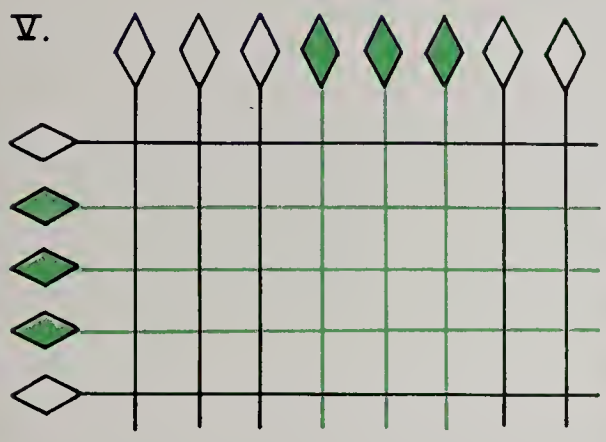
III.



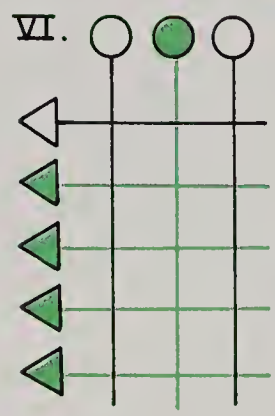
IV.



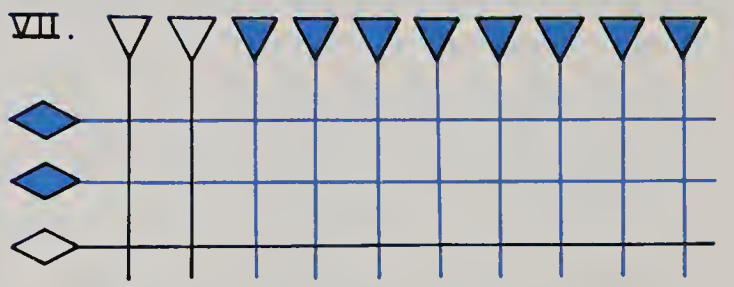
V.



VI.



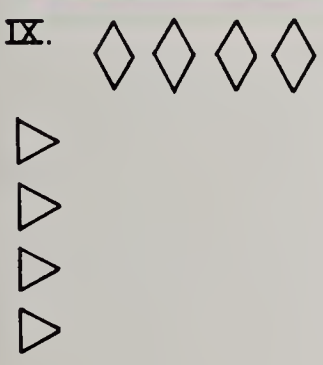
VII.



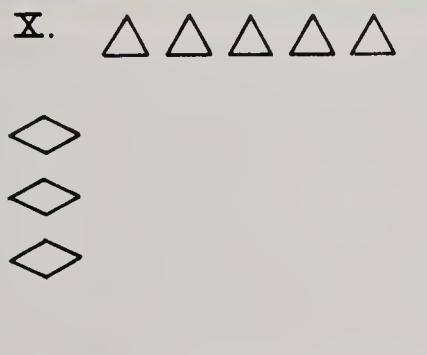
I. — X — = — II. — X — = — III. — X — = — IV. — X — = —

V. — X — = — VI. — X — = — VII. — X — = — VIII. — X — = —

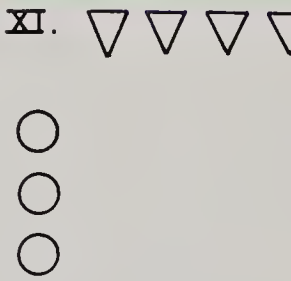
IX.



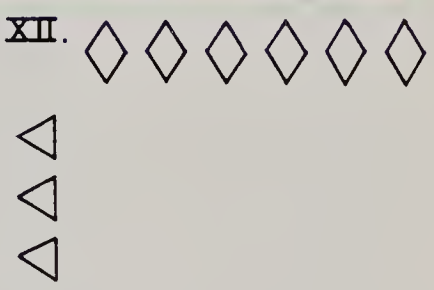
X.



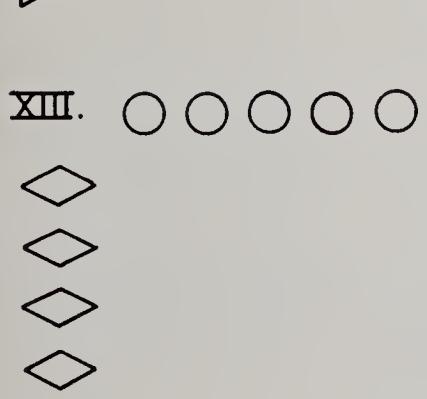
XI.



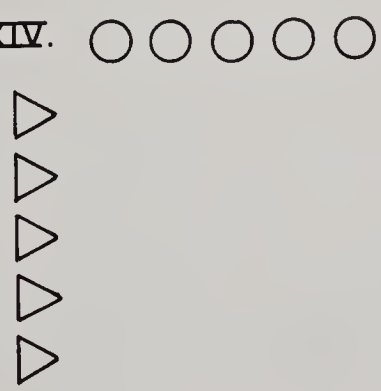
XII.



XIII.

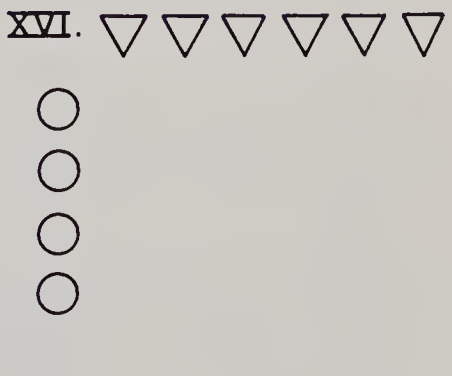


XIV.



XV.

XVI.



IV. $\frac{3}{4} \times \frac{3}{4} = —$ X. $\frac{2}{3} \times \frac{2}{5} = —$ XI. $\frac{2}{3} \times \frac{3}{4} = —$ XII. $\frac{2}{3} \times \frac{5}{6} = —$

XIII. $\frac{3}{4} \times \frac{2}{5} = —$ XIV. $\frac{4}{5} \times \frac{4}{5} = —$ XV. $\frac{3}{5} \times \frac{2}{2} = —$ XVI. $\frac{4}{4} \times \frac{5}{6} = —$

Largest number of factors (other than 1)

3 and 4 are factors of 12. But 4 can be written as a pair of factors — 2 and 2. So 12 can be written as a group of three factors: $12 = 2 \times 2 \times 3$. (Let us agree to write the smaller factors first, because order doesn't effect the results in multiplication.)

Product	Can it be factored?	Written as the most factors	Written as other groups of factors other than 1, with smallest factors listed first
12	yes	2 x 2 x 3	2 x 6 ; 3 x 4
8			
18			
27			
24			
30			
20			
23			
35			
32			
36			
41			
45			
48			
49			
60			
64			
72			
79			

Addend			49	0		19			117		466
Addend	18		49		72		57	81		325	
Sum	34	0		91	100	100	100	200	200	500	1000

Jim Dooley said that the "largest number of factors" could be put to use in simplifying fractions. When asked to explain, he put the following on the blackboard:

$\frac{18}{27} = \frac{2 \times (3 \times 3)}{3 \times (3 \times 3)}$ $\quad \frac{2}{3} \times \frac{9}{9} = \frac{2}{3} \times 1 = \frac{2}{3}$ $\quad \frac{20}{45} = \frac{2 \times 2 \times (5)}{3 \times 3 \times (5)} = \frac{4}{9} \times \frac{5}{5} = \frac{4}{9} \times 1 = \frac{4}{9}$

$\frac{30}{36} = \frac{2 \times 3 \times 5}{2 \times 2 \times 3 \times 3} = \frac{(2 \times 3) \times 5}{(2 \times 3) \times 2 \times 3} = \frac{6}{6} \times \frac{5}{6} = 1 \times \frac{5}{6} = \frac{5}{6}$

Let's test Jim Dooley's method.

	Written using largest number of factors	Rearrange factors if necessary	Complete
$\frac{12}{18} =$	$\frac{2 \times 2 \times}{2 \times 3 \times} =$	$\frac{(2 \times)}{(2 \times)} \frac{\times 2}{\times 3} =$	
$\frac{9}{30} =$	_____ =	_____ =	
$\frac{27}{45} =$	_____ =	_____ =	
$\frac{24}{36} =$	_____ =	_____ =	
$\frac{30}{20} =$	_____ =	_____ =	
$\frac{15}{60} =$	_____ =	_____ =	
$\frac{49}{35} =$	_____ =	_____ =	
$\frac{17}{36} =$	_____ =	_____ =	
$\frac{54}{72} =$	_____ =	_____ =	

Jim has such an important idea, it has been given a name: "Reducing fractions to lowest terms."

Sum	4	0	13	22	27	101	16	17	18	25	110	
Addend												Factor
Addend												Factor
	4	0	42	72	50	100	63	72	81	150	1000	Product

Some numbers can be divided evenly by each of a pair of numbers.

Pairs of numbers	Numbers that can be divided evenly by each
2 , 3	
2 , 8	
9 , 6	
6 , 8	
10 , 25	
4 , 10	
6 , 4	
5 , 3	
7 , 11	
12 , 6	

Anne made three observations she thought were sound:

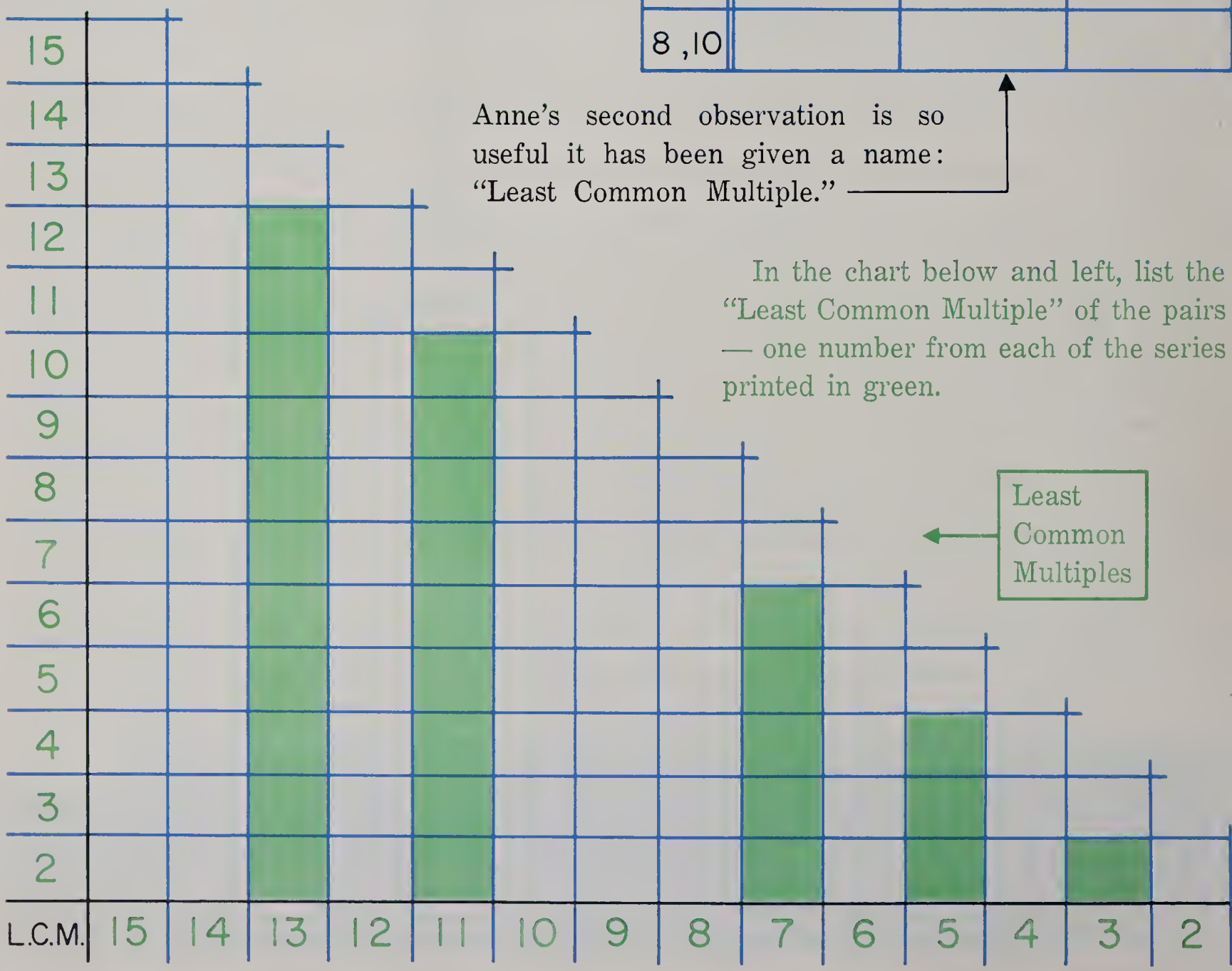
1. "The product of the pair can always be divided by each of the pair."
2. "There is always a smallest number that can be divided by each."
3. "There is no limit to the numbers that can be divided by each."

Let's test these observations:

Pairs	Product	Smallest number?	Other
4 , 8			
6 , 15			
5 , 7			
12 , 4			
8 , 10			

Anne's second observation is so useful it has been given a name: "Least Common Multiple."

In the chart below and left, list the "Least Common Multiple" of the pairs — one number from each of the series printed in green.



Jim Dooley said that the idea of "Least Common Multiple" could help him solve a problem in the addition of fractions. "If I write each fraction with the Least Common Multiple as a denominator, I usually don't have to reduce the result." He put the following examples on the board:

$$\frac{3}{4}+\frac{1}{6}=\frac{18}{24}+\frac{4}{24}=\frac{22}{24}=\frac{11}{12} \qquad \text{but} \qquad \frac{3}{4}+\frac{1}{6}=\frac{9}{12}+\frac{2}{12}=\frac{11}{12}$$

"I could reduce the first result because 24 is not the Least Common Multiple of 4 and 6. And when I use 12 as the denominator and rewrite the addends, I won't be able to reduce the result any further — because 12 is the Least Common Multiple of 6 and 4."

Test Jim's new idea. Use the Least Common Multiple as the denominator indicated by the blocks of blue. Can you reduce any results?

There are some exceptions, such as:

$$\frac{1}{6}+\frac{1}{3}=\frac{3}{6}$$

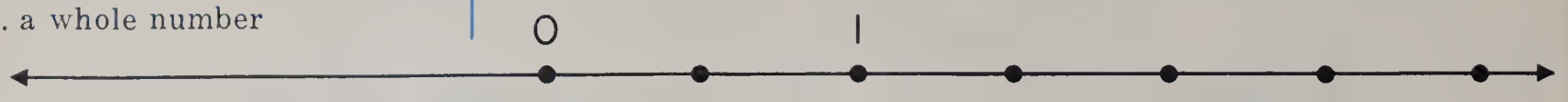
Can you find others?

A. $\frac{1}{6}+\frac{1}{4}=\frac{\square}{\square}+\frac{\square}{\square}=\frac{\square}{\square}$	I. $\frac{3}{10}+\frac{1}{8}=\frac{\square}{\square}+\frac{\square}{\square}=\frac{\square}{\square}$
B. $\frac{3}{4}+\frac{1}{8}=\frac{\square}{\square}+\frac{\square}{\square}=\frac{\square}{\square}$	J. $\frac{9}{20}+\frac{8}{15}=\frac{\square}{\square}+\frac{\square}{\square}=\frac{\square}{\square}$
C. $\frac{1}{9}+\frac{2}{3}=\frac{\square}{\square}+\frac{\square}{\square}=\frac{\square}{\square}$	K. $\frac{5}{14}+\frac{4}{21}=\frac{\square}{\square}+\frac{\square}{\square}=\frac{\square}{\square}$
D. $\frac{1}{8}+\frac{1}{6}=\frac{\square}{\square}+\frac{\square}{\square}=\frac{\square}{\square}$	L. $\frac{3}{10}+\frac{1}{2}=\frac{\square}{\square}+\frac{\square}{\square}=\frac{\square}{\square}$
E. $\frac{3}{8}+\frac{5}{12}=\frac{\square}{\square}+\frac{\square}{\square}=\frac{\square}{\square}$	M. $\frac{2}{4}+\frac{4}{12}=\frac{\square}{\square}+\frac{\square}{\square}=\frac{\square}{\square}$
F. $\frac{1}{9}+\frac{5}{6}=\frac{\square}{\square}+\frac{\square}{\square}=\frac{\square}{\square}$	N. $\frac{1}{3}+\frac{1}{6}=\frac{\square}{\square}+\frac{\square}{\square}=\frac{\square}{\square}$
G. $\frac{3}{16}+\frac{3}{8}=\frac{\square}{\square}+\frac{\square}{\square}=\frac{\square}{\square}$	O. $\frac{3}{7}+\frac{4}{7}=\frac{\square}{\square}+\frac{\square}{\square}=\frac{\square}{\square}$
H. $\frac{2}{5}+\frac{3}{7}=\frac{\square}{\square}+\frac{\square}{\square}=\frac{\square}{\square}$	P. $\frac{1}{3}+\frac{2}{8}=\frac{\square}{\square}+\frac{\square}{\square}=\frac{\square}{\square}$

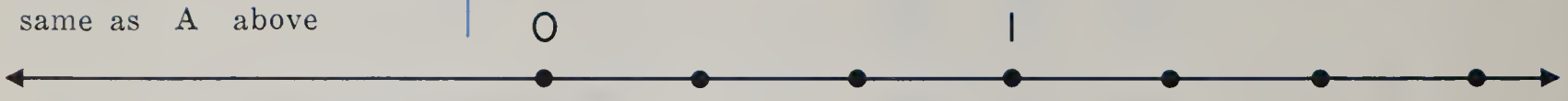
Factor											Addend
Factor											Addend
Product	36	54	56	11	31	29		0	121	1000	
	20	21	18				0	317	22	110	Sum

Rule:
Each label should be:

There are many different ways to label the same number line.



A. a whole number							
B. a fraction with 2 as the denominator	$\frac{0}{2}$						
C. a whole number plus a fraction that is less than 1	$0+\frac{0}{2}$						
D. a whole number minus a fraction that is less than 1	$0-\frac{0}{2}$						
E. a fraction reduced to lowest terms	$\frac{0}{2}$						
F. a fraction not reduced to lowest terms — all with a common denominator	$\frac{0}{6}$						
G. sum of 2 fractions, with 4 as a common denominator							
H. same as C — but leave out fraction equal to 0							



I. same as A above							
J. a fraction with 3 as the denominator							
K. same as C above							
L. same as D above							
M. same as E above							
N. same as K, but leave out as much of the name as you can							
O. same as K, using agreement below							
P. same as N, using agreement below							

Agreement: When a whole number and a fraction are written together, "addition" is implied — and the sign can be omitted.

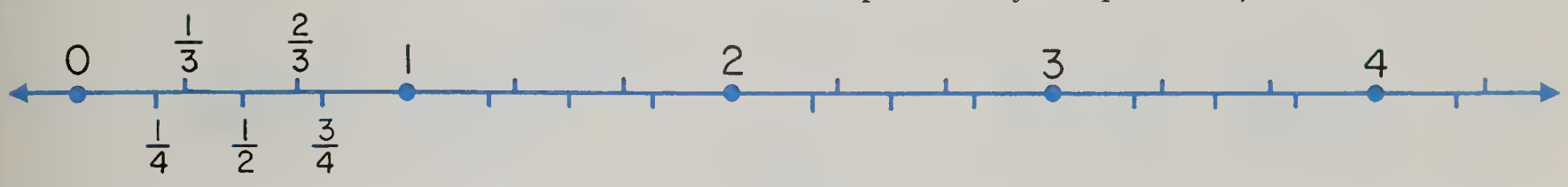
$2+\frac{3}{4}=2\frac{3}{4}$ (Read: "Two and three fourths") $11\frac{5}{9}=11+\frac{5}{9}$

Definition: A number name written as the sum of a whole number and a fraction is called a "MIXED NUMBER"—because it is a mixture of a whole number and a fraction.

Write each of the following as a Mixed Number:

$\frac{3}{2} =$	$\frac{1}{2} + \frac{3}{4} =$	$1 + 1\frac{1}{2} =$	$3 - \frac{3}{4} =$	$1\frac{2}{3} - \frac{1}{3} =$
$\frac{5}{4} =$	$\frac{2}{3} + \frac{2}{3} =$	$4 - \frac{1}{3} =$	$\frac{5}{2} =$	$\frac{11}{3} =$
$\frac{5}{3} =$	$1\frac{1}{2} + 1\frac{1}{4} =$	$1\frac{2}{3} + 1\frac{2}{3} =$	$2\frac{1}{2} - \frac{3}{4} =$	$4\frac{1}{4} - 1\frac{1}{2} =$

(Use the number line below to help answer your questions.)

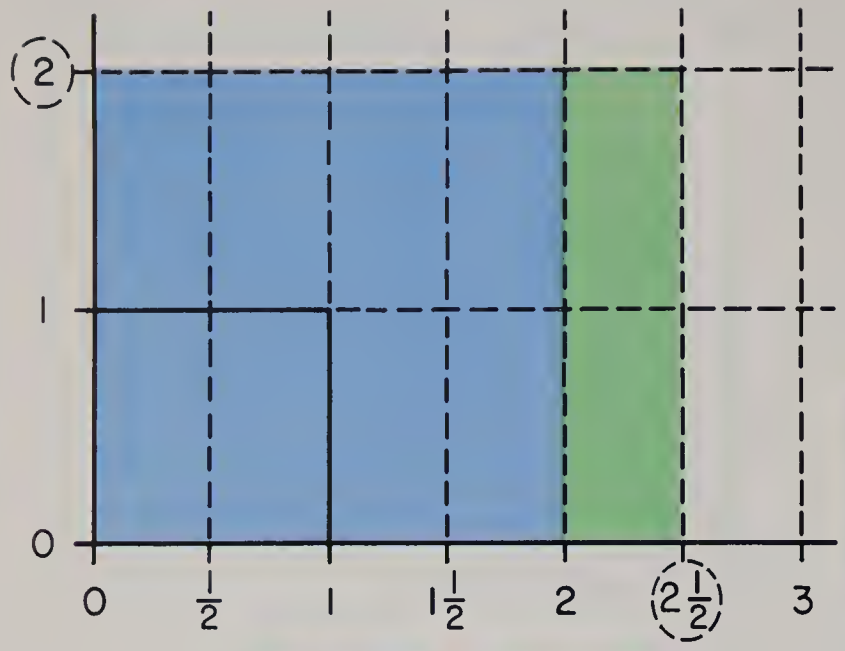


Write each of the following as a whole number plus a fraction, with the fraction to be more than 1 but less than 3:

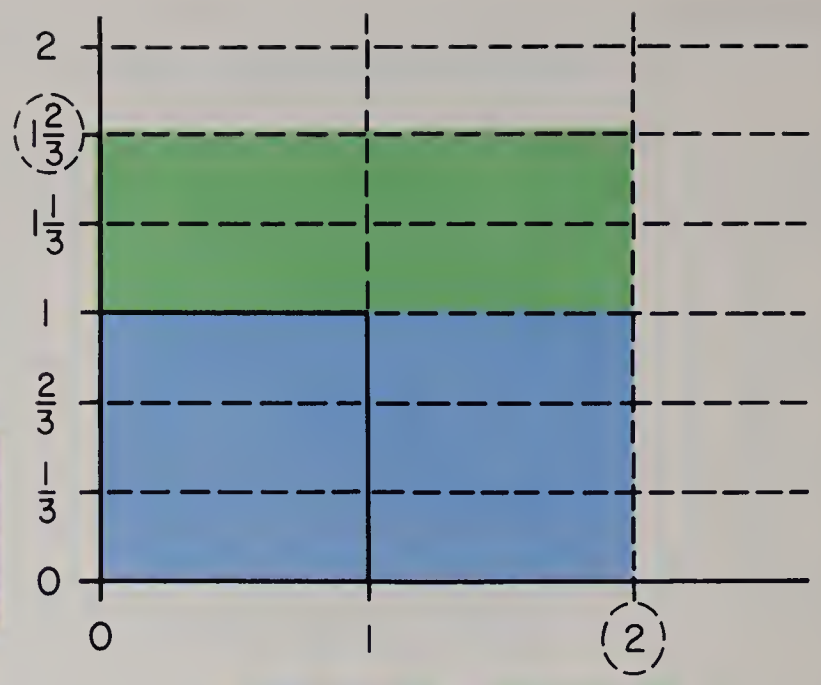
$3\frac{1}{2} = 2 + \frac{}{2}$	$4\frac{2}{3} = 3 + \frac{}{3}$	$2\frac{1}{4} = 1 + \frac{}{4}$	$5\frac{3}{4} = 4 + \frac{}{4}$	$8\frac{1}{5} = 7 + \frac{}{5}$
$3\frac{1}{2} = 1 + \frac{}{2}$	$4\frac{2}{3} = 2 + \frac{}{3}$	$3\frac{1}{4} = 1 + -$	$5\frac{3}{4} = 3 + -$	$8\frac{1}{5} = 6 + \frac{}{5}$
$5\frac{1}{2} = 3 + \frac{}{2}$	$4\frac{1}{3} = 2 + \frac{}{6}$	$5\frac{1}{10} =$	$6\frac{2}{5} =$	$7\frac{3}{8} =$
$4\frac{1}{2} = 3 + \frac{}{2}$	$3\frac{1}{3} = 2 + \frac{}{9}$	$10\frac{2}{3} = 9 + \frac{}{3}$	$10\frac{2}{3} = 9 + \frac{}{6}$	$10\frac{2}{3} = 9 + \frac{}{9}$
$4\frac{1}{2} = 3 + \frac{}{4}$	$5\frac{1}{3} = 4 + \frac{}{9}$	$8\frac{3}{10} =$	$11\frac{1}{12} =$	$3\frac{5}{6} =$
$5\frac{1}{2} = 4 + \frac{}{6}$	$5\frac{1}{3} = 3 + \frac{}{3}$	$5\frac{1}{3} = 3 + \frac{}{9}$	$19\frac{7}{8} =$	$28\frac{1}{11} =$

Write each of the following as the sum of 2 mixed numbers:

$3 = 1\frac{}{2} + 1\frac{}{2}$	$3 = 1\frac{1}{3} +$	$3 = 1\frac{3}{4} +$	$3 = 1\frac{5}{8} +$
$3\frac{1}{2} = 1\frac{1}{4} + 2-$	$3\frac{1}{2} = 1\frac{3}{4} +$	$3\frac{1}{2} = 1\frac{1}{2} + 1\frac{}{2}$	$3\frac{1}{2} = 1\frac{2}{2} +$
$5\frac{1}{3} = 2\frac{2}{3} +$	$5\frac{1}{3} = 2\frac{1}{3} +$	$2\frac{3}{3} = 1\frac{}{3} +$	$2\frac{3}{3} = 1\frac{}{6} +$
$4\frac{1}{5} = 2\frac{}{10} +$	$4\frac{1}{5} = 3\frac{}{15} +$	$2\frac{5}{4} = 1\frac{1}{2} +$	$2\frac{5}{4} = 2- +$
$3\frac{1}{8} = 1- +$	$3\frac{1}{8} = 2- +$	$7\frac{3}{10} = 4- +$	$7\frac{3}{10} = 6- +$



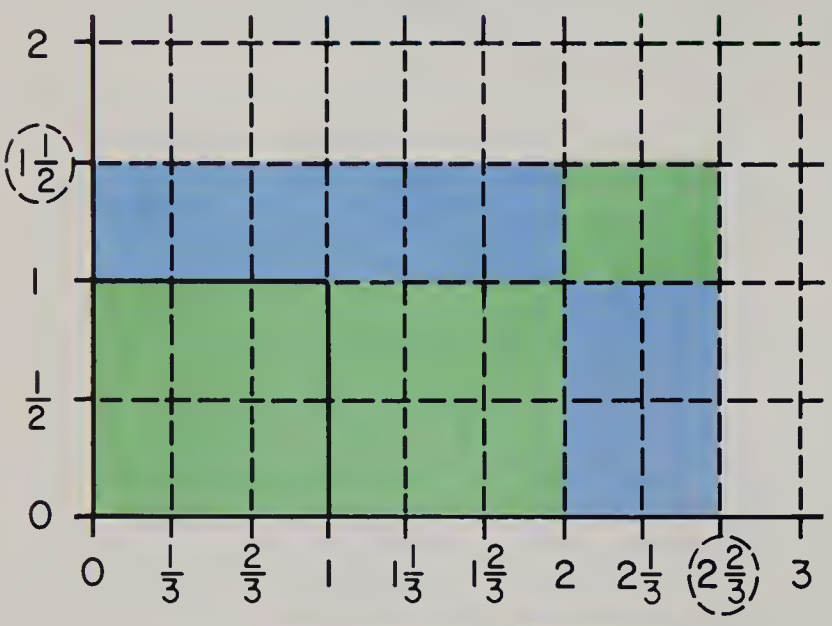
$2\frac{1}{2} \times 2 =$ 2 x 2 + $\frac{1}{2} \times 2$ =



$2 \times 1\frac{2}{3}$ = 2 x + 2 x =

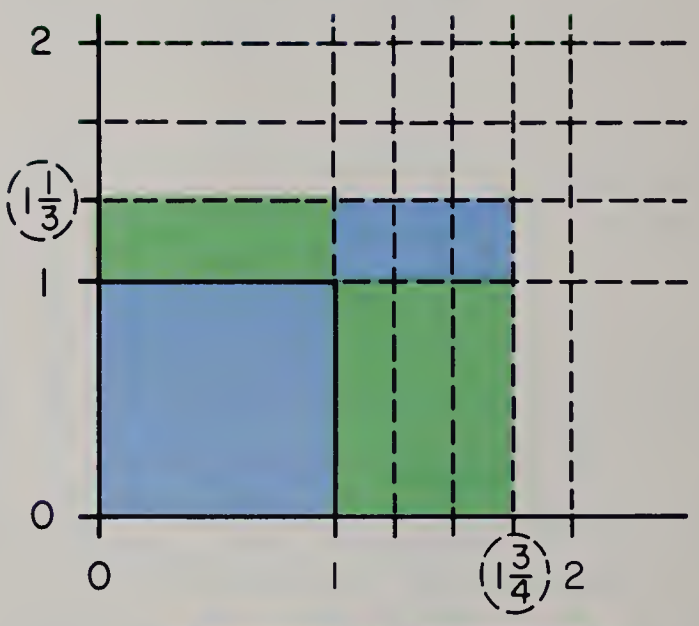
A B

C D



$2\frac{2}{3} \times 1\frac{1}{2} =$ 2 x + $\frac{2}{3} \times$ +

2 x + $\frac{2}{3} \times$ =

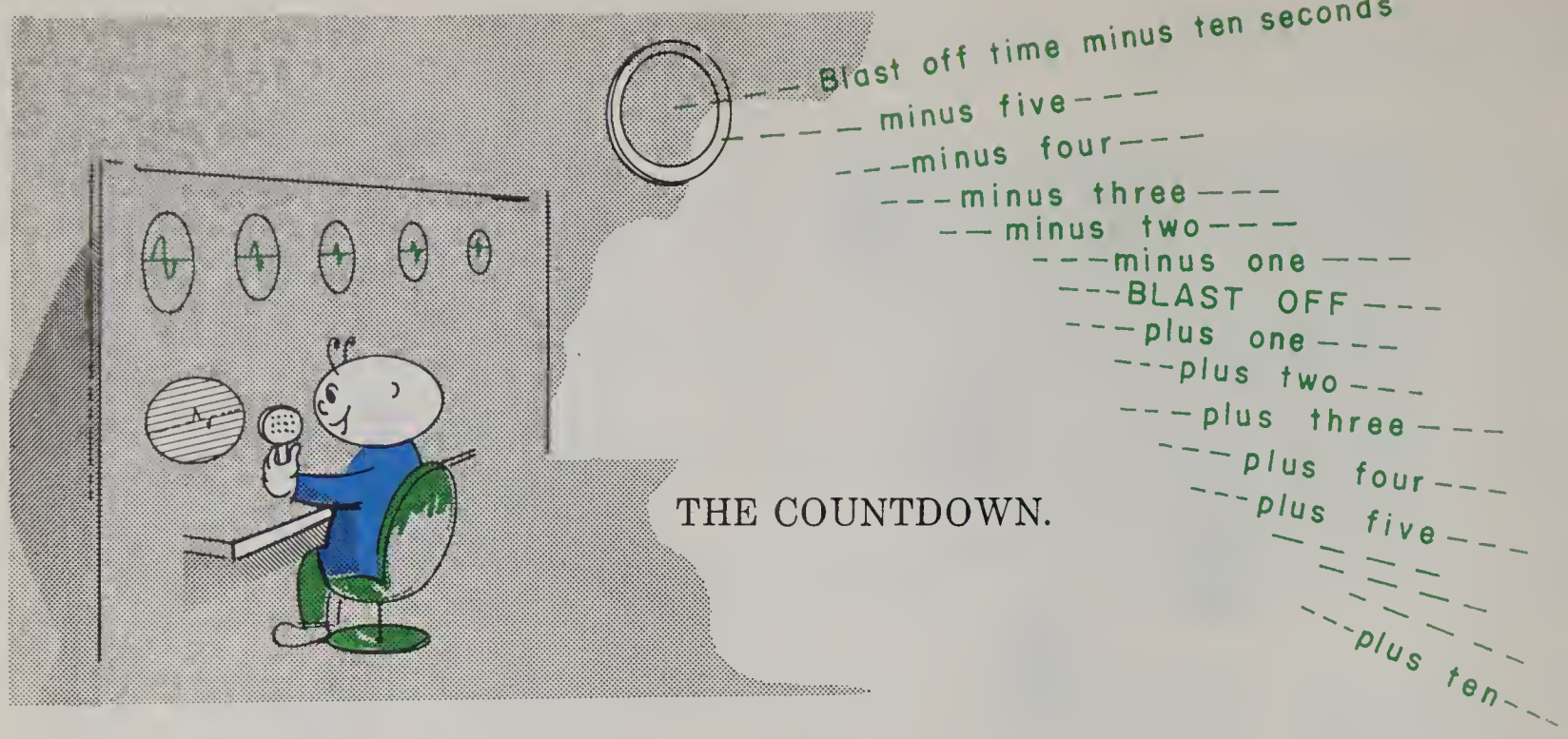


$1\frac{3}{4} \times 1\frac{1}{3} =$ + +

+ =

How far could this chart be extended?

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
18	1/18	2/18	3/18	4/18	5/18	6/18	7/18	8/18	9/18	10/18	11/18	12/18	13/18	14/18	15/18	16/18	17/18	18/18
17	1/17	2/17	3/17	4/17	5/17	6/17	7/17	8/17	9/17	10/17	11/17	12/17	13/17	14/17	15/17	16/17	17/17	18/17
16	1/16	2/16	3/16	4/16	5/16	6/16	7/16	8/16	9/16	10/16	11/16	12/16	13/16	14/16	15/16	16/16	17/16	18/16
15	1/15	2/15	3/15	4/15	5/15	6/15	7/15	8/15	9/15	10/15	11/15	12/15	13/15	14/15	15/15	16/15	17/15	18/15
14	1/14	2/14	3/14	4/14	5/14	6/14	7/14	8/14	9/14	10/14	11/14	12/14	13/14	14/14	15/14	16/14	17/14	18/14
13	1/13	2/13	3/13	4/13	5/13	6/13	7/13	8/13	9/13	10/13	11/13	12/13	13/13	14/13	15/13	16/13	17/13	18/13
12	1/12	2/12	3/12	4/12	5/12	6/12	7/12	8/12	9/12	10/12	11/12	12/12	13/12	14/12	15/12	16/12	17/12	18/12
11	1/11	2/11	3/11	4/11	5/11	6/11	7/11	8/11	9/11	10/11	11/11	12/11	13/11	14/11	15/11	16/11	17/11	18/11
10	1/10	2/10	3/10	4/10	5/10	6/10	7/10	8/10	9/10	10/10	11/10	12/10	13/10	14/10	15/10	16/10	17/10	18/10
9	1/9	2/9	3/9	4/9	5/9	6/9	7/9	8/9	9/9	10/9	11/9	12/9	13/9	14/9	15/9	16/9	17/9	18/9
8	1/8	2/8	3/8	4/8	5/8	6/8	7/8	8/8	9/8	10/8	11/8	12/8	13/8	14/8	15/8	16/8	17/8	18/8
7	1/7	2/7	3/7	4/7	5/7	6/7	7/7	8/7	9/7	10/7	11/7	12/7	13/7	14/7	15/7	16/7	17/7	18/7
6	1/6	2/6	3/6	4/6	5/6	6/6	7/6	8/6	9/6	10/6	11/6	12/6	13/6	14/6	15/6	16/6	17/6	18/6
5	1/5	2/5	3/5	4/5	5/5	6/5	7/5	8/5	9/5	10/5	11/5	12/5	13/5	14/5	15/5	16/5	17/5	18/5
4	1/4	2/4	3/4	4/4	5/4	6/4	7/4	8/4	9/4	10/4	11/4	12/4	13/4	14/4	15/4	16/4	17/4	18/4
3	1/3	2/3	3/3	4/3	5/3	6/3	7/3	8/3	9/3	10/3	11/3	12/3	13/3	14/3	15/3	16/3	17/3	18/3
2	1/2	2/2	3/2	4/2	5/2	6/2	7/2	8/2	9/2	10/2	11/2	12/2	13/2	14/2	15/2	16/2	17/2	18/2
1	1/1	2/1	3/1	4/1	5/1	6/1	7/1	8/1	9/1	10/1	11/1	12/1	13/1	14/1	15/1	16/1	17/1	18/1
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18



THE COUNTDOWN.

"Blast off minus one minute," came over the loud speaker.

"Blast off minus 30 seconds."

Bill looked out of the window and thought, "Well, Mr. Moon, here we come!"

"Blast off minus 10 seconds . . . minus 5 seconds . . . minus 4, minus 3, minus 2, minus 1 . . . Blast off!"

"Blast off plus 1 . . . plus 2 . . . plus 3 . . ."

As the count went on, Bill looked out again to see the earth falling away below.

When the announcer reached ". . . plus 30 seconds . . ." he stopped counting and shouted happily: "Everything is A. O. K."

What is counting like before "Blast off?"
What is counting like after "Blast off?"

How long is it between two calls?

1st Call	2 nd Call	Seconds Between
minus 5	minus 3	
minus 7	BLAST OFF	
minus 6	minus 1	
BLAST OFF	plus 4	
minus 1	plus 1	
minus 5	plus 5	
minus 3	plus 4	
minus 8	minus 2	

Think of other examples in which you count in two directions from a known point.

"I counted the money in the cash register this morning. I just counted it now, and there is $7.00 difference."

"More or less?"

"Less. I made a large refund and a small sale," clerk Jones answered.

"I'm five miles from you on Route 27," Mr. Brown told the tow-truck operator over the phone. "Will you come help me?"

"Sure. Five miles toward Yorkville?"

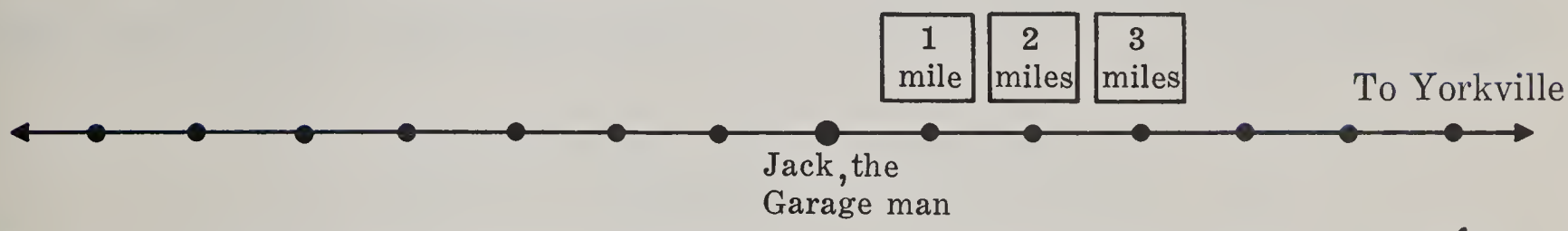
"No, Jack," Mr. Brown replied. "Five miles in the opposite direction."

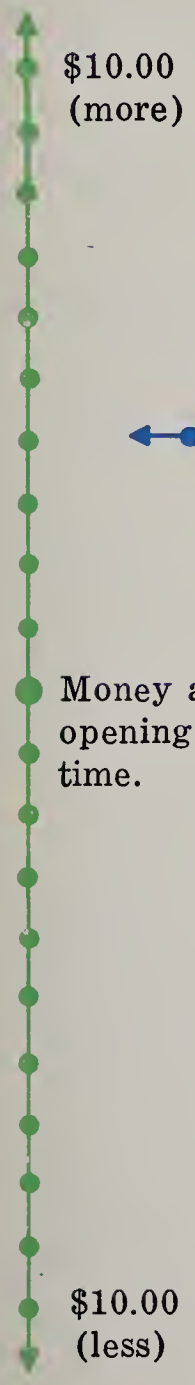
Make up your own stories where we ask such questions as:

Up or down?
Above or below?
Swelling or shrinking?
Ahead or behind?

Faster or slower?
Forward or backward?
Overweight or underweight?
Before or after?

North or South?
East or West?
Profit or Loss?
Stretch or contract?

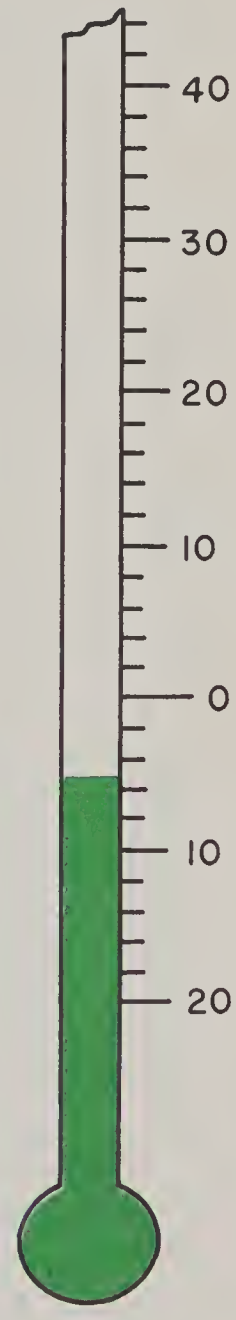




A. On the map above, mark an × to show the place where Mr. Brown had trouble.

B. On the chart at left, mark an × to show the amount of money clerk Jones had at the last count.

C. In the picture on the right of part of a thermometer, can you tell the temperature?



We read "2 ↶ 1 " as "two and a jump of one to the left" takes us to 1 or $2 - 1 \rightarrow$ ___

Draw a picture to suggest 5 ↶ 2 or $5 - 2 \rightarrow$ ___

Draw pictures that suggest the following:

2 ↶ 3 or $2 - 3 \rightarrow$ ___

0 ↶ 5 or $0 - 5 \rightarrow$ ___

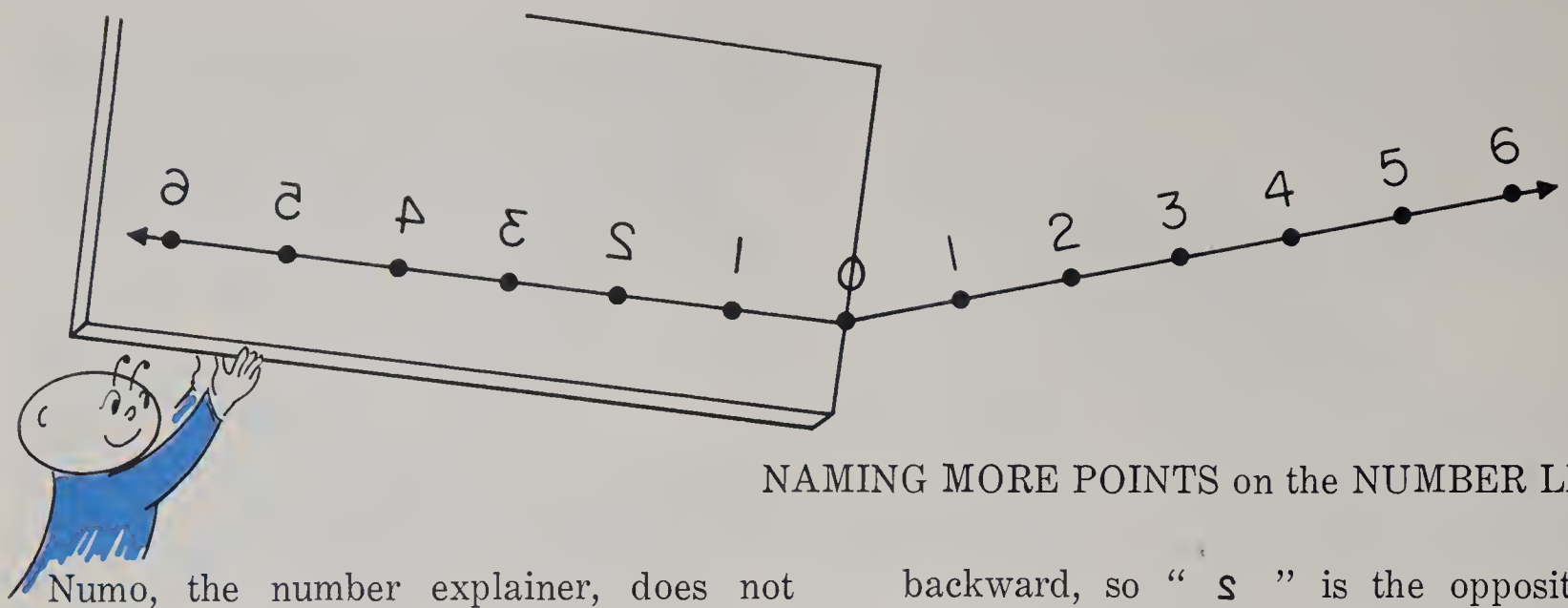
3 ↶ 5 or $3 - 5 \rightarrow$ ___

To complete the above three statements, label the points on the line on the side of 0 opposite the blue numeral:

① , ② , ③ , ④ , ⑤ , etc.

Read them as "①" is the opposite of " 1, " "②" is the opposite of " 2, " etc.

2 ↶ 2 → ___ 2 ↶ 4 → ___ 2 ↶ 7 → ___ 2 ↶ 5 → ___



NAMING MORE POINTS on the NUMBER LINE.

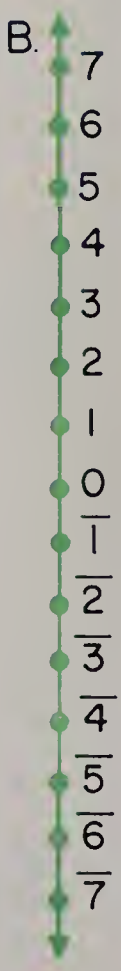
Numo, the number explainer, does not always have his best ideas first. He is suggesting that we learn to write numbers backward, so " ƽ " is the opposite of " 2, " and " Ɛ " is the opposite of " 3. " Is there anything wrong with this idea?

We are going to write "the opposite of 3" as "$\overline{3}$," and "the opposite of 1" as "$\overline{1}$."

This mark, "−" to mean "the opposite of" is suggested by the ". . . minus . . ." in the countdown. It is also suggested by ". . . below . . ." in reading temperatures.

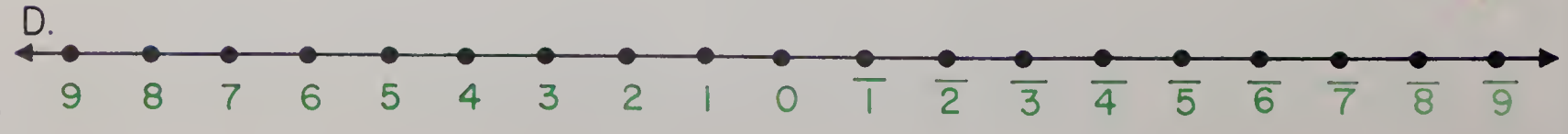
A.

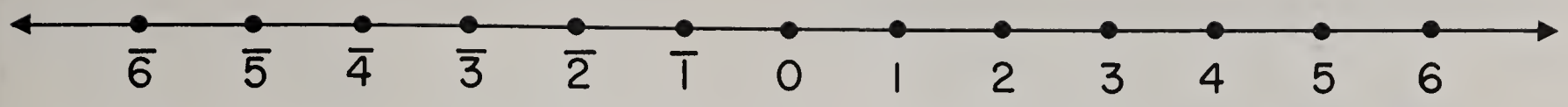
We usually draw number lines either horizontally or vertically on the page and list the counting number to the right or above 0. But this is not necessary. Draw pictures to suit the following statements on any line you choose.



1. 2 − 1 → ___
2. 2 − 2 → ___
3. 2 − 3 → ___
4. 2 − 4 → ___
5. 2 − 2 → ___
6. 2 − 1 → ___
7. 2 − 0 → ___
8. 2 − $\overline{1}$ → ___
9. 1 + 1 → ___
10. 0 + 1 → ___
11. $\overline{1}$ + 1 → ___
12. $\overline{2}$ + 1 → ___
13. $\overline{3}$ + 1 → ___
14. $\overline{1}$ + 1 → ___
15. $\overline{1}$ + 0 → ___
16. $\overline{1}$ + $\overline{1}$ → ___
17. 2 x 2 → ___
18. 2 x 1 → ___
19. 2 x 0 → ___
20. 2 x $\overline{1}$ → ___
21. 2 x $\overline{2}$ → ___
22. 1 x 2 → ___
23. 0 x 2 → ___
24. $\overline{1}$ x 2 → ___

D.





A	B	C	D	E
$3+1 \rightarrow \_\_$	$3+2 \rightarrow \_\_$	$4+2 \rightarrow \_\_$	$5-3 \rightarrow \_\_$	$4-2 \rightarrow \_\_$
$2+1 \rightarrow \_\_$	$3+1 \rightarrow \_\_$	$3+1 \rightarrow \_\_$	$4-3 \rightarrow \_\_$	$3-2 \rightarrow \_\_$
$1+1 \rightarrow \_\_$	$3+0 \rightarrow \_\_$	$2+0 \rightarrow \_\_$	$3-3 \rightarrow \_\_$	$2-2 \rightarrow \_\_$
$0+1 \rightarrow \_\_$	$3+\bar{1} \rightarrow \_\_$	$1+\bar{1} \rightarrow \_\_$	$2-3 \rightarrow \_\_$	$1-2 \rightarrow \_\_$
$\bar{1}+1 \rightarrow \_\_$	$3+\bar{2} \rightarrow \_\_$	$0+\bar{2} \rightarrow \_\_$	$1-3 \rightarrow \_\_$	$0-2 \rightarrow \_\_$
$\bar{2}+1 \rightarrow \_\_$	$3+\bar{3} \rightarrow \_\_$	$\bar{1}+\bar{3} \rightarrow \_\_$	$0-3 \rightarrow \_\_$	$\bar{1}-2 \rightarrow \_\_$
$\bar{3}+1 \rightarrow \_\_$	$3+\bar{4} \rightarrow \_\_$	$\bar{2}+\bar{4} \rightarrow \_\_$	$\bar{1}-3 \rightarrow \_\_$	$\bar{2}-2 \rightarrow \_\_$
$\bar{4}+1 \rightarrow \_\_$	$3+\bar{5} \rightarrow \_\_$	$\bar{3}+\bar{5} \rightarrow \_\_$	$\bar{2}-3 \rightarrow \_\_$	$\bar{3}-2 \rightarrow \_\_$

F	G	H	T	J
$2-4 \rightarrow \_\_$	$\bar{3}-2 \rightarrow \_\_$	$\bar{3}+3 \rightarrow \_\_$	$3+2 \rightarrow \_\_$	$2+3 \rightarrow \_\_$
$2-3 \rightarrow \_\_$	$\bar{3}-1 \rightarrow \_\_$	$\bar{2}+2 \rightarrow \_\_$	$\bar{3}+2 \rightarrow \_\_$	$2+\bar{3} \rightarrow \_\_$
$2-2 \rightarrow \_\_$	$\bar{3}-0 \rightarrow \_\_$	$\bar{1}+1 \rightarrow \_\_$	$3+\bar{2} \rightarrow \_\_$	$\bar{2}+3 \rightarrow \_\_$
$2-1 \rightarrow \_\_$	$\bar{3}-\bar{1} \rightarrow \_\_$	$0+0 \rightarrow \_\_$	$\bar{3}+\bar{2} \rightarrow \_\_$	$\bar{2}+\bar{3} \rightarrow \_\_$
$2-0 \rightarrow \_\_$	$\bar{3}-\bar{2} \rightarrow \_\_$	$1+\bar{1} \rightarrow \_\_$	$3-2 \rightarrow \_\_$	$2-3 \rightarrow \_\_$
$2-\bar{1} \rightarrow \_\_$	$\bar{3}-\bar{3} \rightarrow \_\_$	$2+\bar{2} \rightarrow \_\_$	$\bar{3}-2 \rightarrow \_\_$	$\bar{2}-3 \rightarrow \_\_$
$2-\bar{2} \rightarrow \_\_$	$\bar{3}-\bar{4} \rightarrow \_\_$	$3+\bar{3} \rightarrow \_\_$	$3-\bar{2} \rightarrow \_\_$	$2-\bar{3} \rightarrow \_\_$
$2-\bar{3} \rightarrow \_\_$	$\bar{3}-\bar{5} \rightarrow \_\_$	$4+\bar{4} \rightarrow \_\_$	$\bar{3}-\bar{2} \rightarrow \_\_$	$\bar{2}-\bar{3} \rightarrow \_\_$
$2-\bar{4} \rightarrow \_\_$	$\bar{3}-\bar{6} \rightarrow \_\_$	$5+\bar{5} \rightarrow \_\_$	$3-\bar{2} \rightarrow \_\_$	$3+2 \rightarrow \_\_$

New NAMES for $\bar{2}$	New NAMES for $\bar{1}$	New NAMES for 0	New NAMES for 1	New NAMES for 2

Now you can play the Game of OPPOSITES

Here is the playing board:

Mary

	A	B	C
X	1	$\bar{2}$	1
Y	0	1	$\bar{1}$
Z	$\bar{1}$	1	0

Jim

Jim and Mary are playing.

Each turn, Jim picks a letter on his side of the board (A, B, or C.) He writes it down. Mary cannot look at Jim's choice.

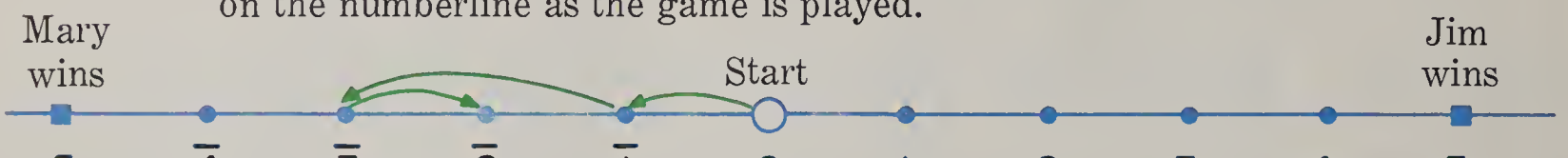
Then Mary picks a letter on her side of the board (X, Y, or Z).

The score for the turn is found where the row picked by Jim and the column picked by Mary cross.

Example: Jim chooses A, and Mary chooses Z. The score is $\bar{1}$ — "the opposite of one."

Keeping Score

Put a marker on the starting point. Mary wins if the counter reaches $\bar{5}$. Jim wins if the counter reaches 5. Move the counter on the numberline as the game is played.



The score, after the example, is $\bar{1}$. Who is ahead?

On the next turn, Jim chooses B, and Mary chooses X. (B, X → $\bar{2}$)

Now the total score is $\bar{3}$. Who is ahead?

On the next play, Jim chooses A, and Mary chooses X again. (A, X → 1)

Now the total score is $\bar{2}$.

That helped Jim, but Mary is still ahead.

J	M	TS	S
			0
A	Z	$\bar{1}$	$\bar{1}$
B	X	$\bar{2}$	$\bar{3}$
A	X	1	$\bar{2}$
C	Z		
Y	C		
Y	B		
B	X		
C	Y		

(M means Mary; J — Jim; TS — Turn Score; S — Total Score.)

What was the total score at the end?

You don't need a score pad to play this game.

I. 0 —AX→ 1 —AZ→ 0 —BZ→ 1 —BX→ $\bar{1}$ —AZ→ $\bar{2}$ —CY→ $\bar{3}$ —BX→ $\bar{5}$ END

II. 0 —CX→ 1 —BZ→ 2 —AX→ 3 —AZ→ 2 —BY→ 3 —BZ→ 4 —AX→ 5 END

III. 0 —CZ→ ___ —BY→ ___ —BX→ ___ —BX→ ___ —AZ→ ___ —AX→ ___ —BX→ ___

IV. 0 —BX→ ___ —AX→ ___ —CX→ ___ —BY→ ___ —BZ→ ___ —CZ→ ___ —CX→ ___ (cont.)

—BZ→ ___ —BX→ ___ —AX→ ___ —BZ→ ___ —AY→ ___ —AZ→ ___ —BZ→ ___ —BY→ ___ END

Scotty Brown thought this game was too slow, so he invented

The Game of Double Opposites.

Scotty's Rule: Always double the turn score.

That's the only change. Here are Sample games.

1. 0 $\xrightarrow{CY}$ $\bar{2}$ $\xrightarrow{AZ}$ ___ $\xrightarrow{BY}$ ___ $\xrightarrow{BX}$ ___ – END

2. 0 $\xrightarrow{BZ}$ ___ $\xrightarrow{BY}$ ___ $\xrightarrow{CZ}$ ___ $\xrightarrow{CX}$ ___ – END

Ellen Burke wanted to speed the game up in another way so she invented

The Short Game of Opposites

Ellen's Rule: The player that reaches 3 or $\bar{3}$ first, wins.

Select the plays so the games end as shown.

3. 0 → ___ → ___ → ___ → ___ – END

4. 0 → ___ → ___ – END

Make up your own rules for your GAME of OPPOSITES.

Complete the headlines and check one that best fits each story.

1. Bill thought: "We are at 3 minutes before blast-off time. Where will we be in 5 minutes?"

$\bar{3} + \bar{5} \rightarrow$ ___	$\bar{3} + 5 \rightarrow$ ___
$\bar{3} +$ ___ $\rightarrow 5$	___ $+ \bar{3} \rightarrow \bar{5}$

2. I lost a nickel and found a dime.

$\bar{5} +$ ___ $\rightarrow 10$	$5 - 10 \rightarrow$ ___
$\bar{5} + \bar{10} \rightarrow$ ___	$\bar{5} + 10 \rightarrow$ ___

3. After I opened the store, Mrs. Jones returned a $4.00 book and bought a $3.00 book instead.

$\bar{3} +$ ___ $\rightarrow \bar{1}$	$\bar{4} +$ ___ $\rightarrow \bar{1}$
$\bar{4} + \bar{3} \rightarrow$ ___	___ $+ \bar{3} \rightarrow 7$

4. The temperature was 5 degrees below zero and fell to 8 degrees below zero.

$\bar{5} + \bar{8} \rightarrow$ ___	$\bar{5} -$ ___ $\rightarrow \bar{8}$
$\bar{5} - 8 \rightarrow$ ___	$5 - 8 \rightarrow$ ___

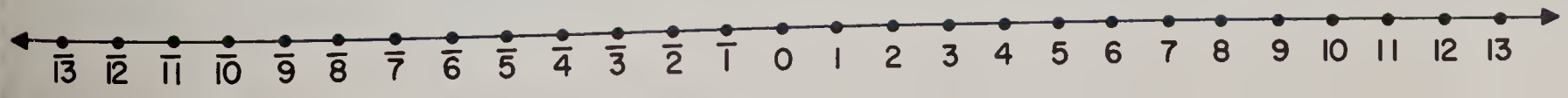
5. Meg and Polly each lived 4 blocks from school, but in opposite directions.

$\bar{4} + \bar{4} \rightarrow$ ___	$2 - 4 \rightarrow$ ___
$\bar{4} +$ ___ $\rightarrow 4$	$8 + \bar{8} \rightarrow$ ___

6. Jerry dug a hole 10 feet deep. He put a 12 foot ladder in the hole. How much of the ladder was above ground.

$\bar{10} +$ ___ $\rightarrow 2$	$10 + 12 \rightarrow$ ___
$\bar{10} + \bar{12} \rightarrow$ ___	$\bar{12} +$ ___ $\rightarrow 2$

Use the number line below to help picture the situations above.



The rule for these Cross Number Puzzles is ADDITION

$\bar{3} + 1 \rightarrow$ ___

$2 + 1 \rightarrow$ ___

$\bar{2} + 3 \rightarrow$ ___

$\bar{3} + 2 \rightarrow$ ___

$1 + 1 \rightarrow$ ___

$\bar{1} + 2 \rightarrow$ ___

$1 + \bar{3} \rightarrow$ ___

$\bar{1} + 2 \rightarrow$ ___

$\bar{2} + 1 \rightarrow$ ___

$1 + \bar{1} \rightarrow$ ___

$\bar{3} + 2 \rightarrow$ ___

$0 + \bar{1} \rightarrow$ ___

A

$\bar{3}$	1	
2	1	

B

1	$\bar{3}$	
$\bar{1}$	2	

$0 + \bar{3} \rightarrow$ ___

$\bar{1} + 2 \rightarrow$ ___

___ + ___ $\rightarrow$ ___

$0 + \bar{1} \rightarrow$ ___

$\bar{3} + 2 \rightarrow$ ___

___ + ___ $\rightarrow$ ___

___ + 5 $\rightarrow \bar{3}$

___ + ___ $\rightarrow$ ___

$\bar{3} +$ ___ $\rightarrow \bar{2}$

___ + ___ $\rightarrow$ ___

$\bar{5} +$ ___ $\rightarrow \bar{1}$

___ + $\bar{1} \rightarrow \bar{2}$

C

0	$\bar{3}$	
$\bar{1}$	2	

D

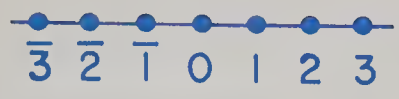
	$\bar{5}$	$\bar{3}$
	$\bar{1}$	$\bar{2}$

The RULE for these Cross Number Puzzles is SUBTRACTION.

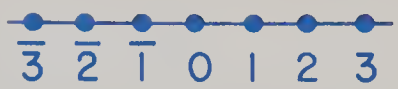
I

2	4	
1	2	

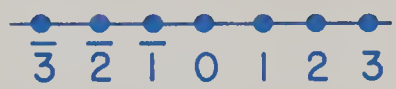
2 − 4 → ___



1 − 2 → ___



___ − ___ → ___



2 − 1 → ___

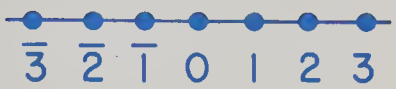
4 − 2 → ___

___ − ___ → ___

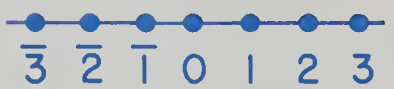
II

3	6	
1	4	

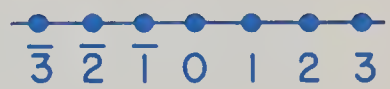
3 − 6 → ___



1 − 4 → ___



___ − ___ → ___



3 − 1 → ___

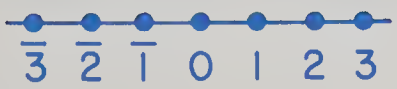
6 − 4 → ___

___ − ___ → ___

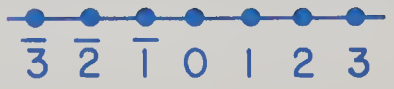
III

0	$\bar{1}$	
$\bar{3}$	$\bar{1}$	

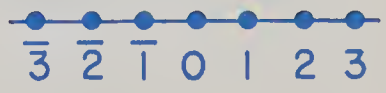
0 − $\bar{1}$ → ___



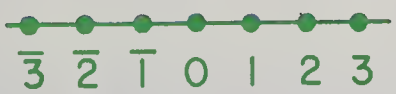
$\bar{3}$ − $\bar{1}$ → ___



___ − ___ → ___

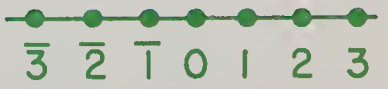


0 − $\bar{3}$ → ___



$\bar{1}$ − $\bar{1}$ → ___

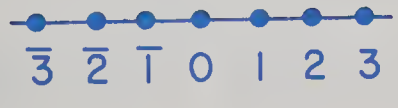
___ − ___ → ___



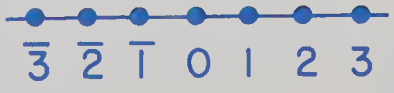
IV

1	2	
3	5	

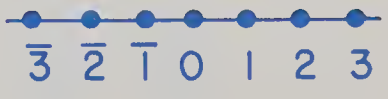
1 − 2 → ___



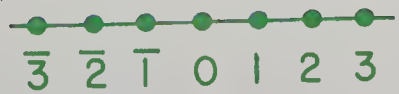
3 − 5 → ___



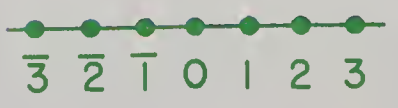
___ − ___ → ___



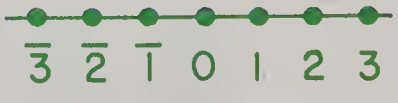
1 − 3 → ___

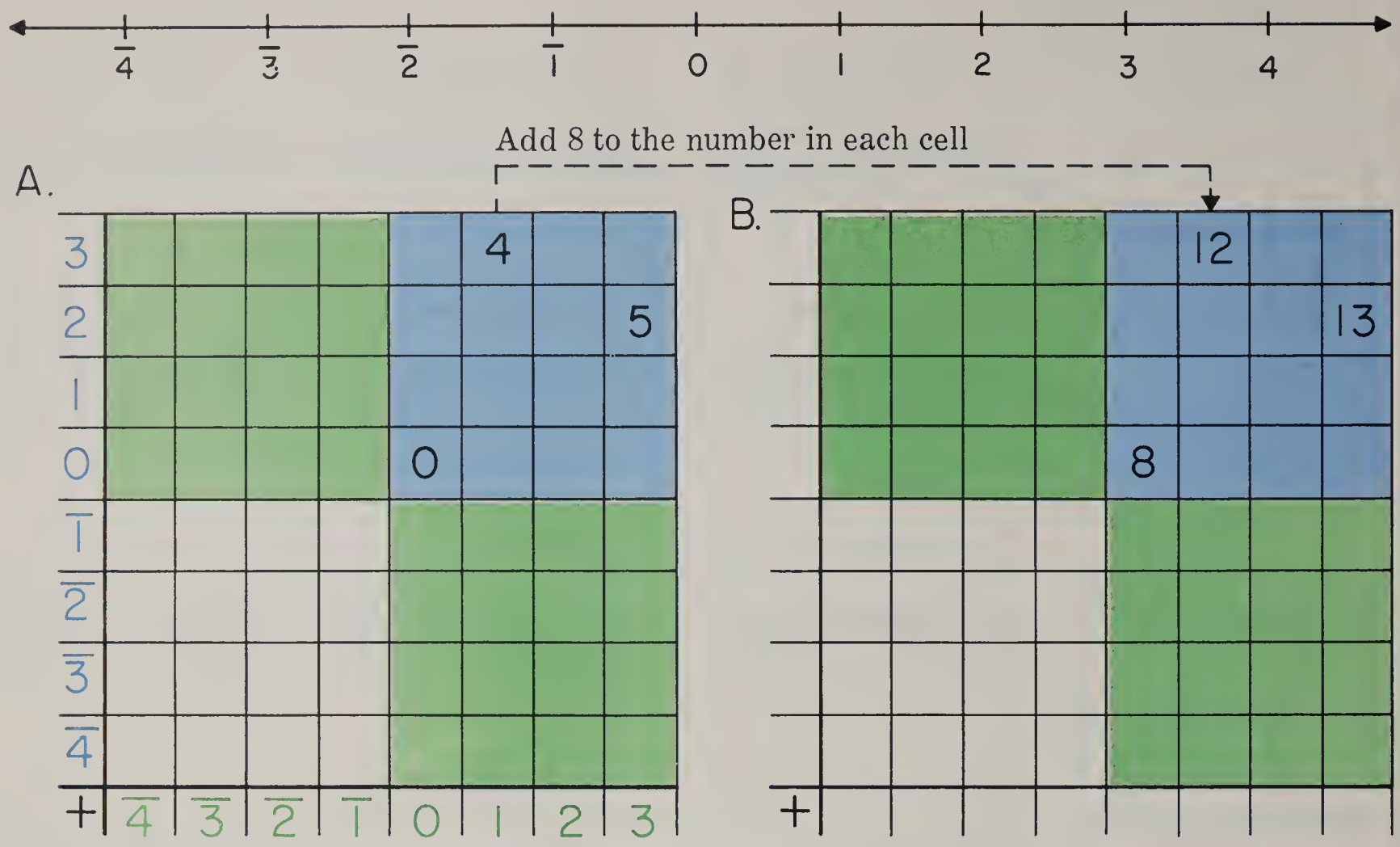


2 − 5 → ___



___ − ___ → ___




$\overline{4}$ $\overline{3}$ $\overline{2}$ $\overline{1}$ 0 1 2 3 4
Add 8 to the number in each cell
A.
B.
3 2 1 0 $\overline{1}$ $\overline{2}$ $\overline{3}$ $\overline{4}$
+ $\overline{4}$ $\overline{3}$ $\overline{2}$ $\overline{1}$ 0 1 2 3
4 5 0
12 13 8
+

	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r
Addend	1	0	$\overline{1}$	$\overline{2}$	$\overline{3}$			$\overline{1}$	1	3	5	7		$\overline{3}$		1
Addend	2	2	2	2	2	2	2	$\overline{3}$	$\overline{3}$	$\overline{3}$	$\overline{3}$	$\overline{3}$	$\overline{5}$		$\overline{1}$	
Sum						$\overline{2}$	$\overline{3}$						$\overline{5}$	0	1	$\overline{2}$

s. If 3 + 5 = 8, then 5 + ▢ = 8 and 8 − ▢ = 5 and 8 − ▢ = ▢

t. If $\overline{2}$ + 3 = ▢, then ▢ + ▢ = ▢ and ▢ − ▢ = ▢ and ▢ − ▢ = ▢

u. If $\overline{1}$ + $\overline{4}$ = ▢, then ▢ + ▢ = ▢ and ▢ − ▢ = ▢ and ▢ − ▢ = ▢

v. If 6 + $\overline{8}$ = ▢, then ▢ + ▢ = ▢ and ▢ − ▢ = ▢ and ▢ − ▢ = ▢

w. If $\overline{3}$ + $\overline{2}$ = ▢, then ▢ + ▢ = ▢ and ▢ − ▢ = ▢ and ▢ − ▢ = ▢

x. If $\overline{5}$ + 7 = ▢, then ▢ + ▢ = ▢ and ▢ − ▢ = ▢ and ▢ − ▢ = ▢

Someday you will find out that there are times when we want to multiply numbers and opposites of numbers. You might be amused by this page that takes a look at this idea.

Think of this as a multiplying machine. Fill in the products that belong in the areas shaded blue.

3						6	
2							
1						2	
0				0			
X				0	1	2	3

C B A

ADD 9
to the number
in each cell

Extend the patterns into the area shaded green.

						15	18
						11	
				9	9		
X							

Fill in the remaining cells like the chart below. Extend the patterns in the left column and bottom row.

SUBTRACT 9
from the number
in each cell.

3						6	
2							
1						2	
0				0			
X				0	1	2	3

B

A

$\bar{1} \times 3 =$ ___
$\bar{1} \times 2 =$ ___
$\bar{1} \times 1 =$ ___
$\bar{1} \times 0 =$ ___
$\bar{1} \times \bar{1} =$ ___
$\bar{1} \times \bar{2} =$ ___
$\bar{1} \times \bar{3} =$ ___

C

$\bar{3} \times 3 =$ ___
$\bar{3} \times 2 =$ ___
$\bar{3} \times 1 =$ ___
$\bar{3} \times 0 =$ ___
$\bar{3} \times \bar{1} =$ ___
$\bar{3} \times \bar{2} =$ ___
$\bar{3} \times \bar{3} =$ ___

Factor	3	2	1	0	$\bar{1}$	$\bar{2}$	$\bar{3}$
Factor	$\bar{2}$	$\bar{2}$	$\bar{2}$	$\bar{2}$	$\bar{2}$	$\bar{2}$	$\bar{2}$
Product							

Make the SENTENCES TRUE

$=$ means "equal to" $<$ means "less than" $>$ means "greater than"

Examples: $3 = 3$ $6 < 10$ $7 > 4$

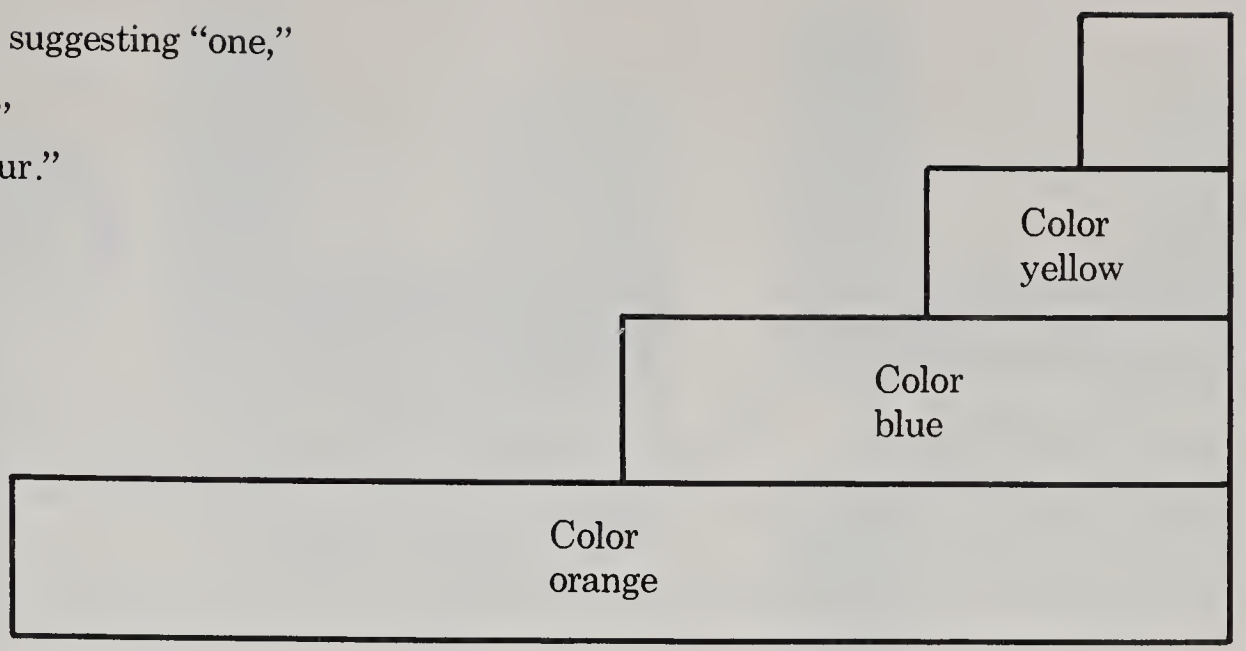
$6 \div 3 \quad \frac{4}{2}$	$2 \div 1 \quad 1 \cdot 2$	$\frac{3}{1} \quad 3 \cdot 1$
$\frac{9}{3} \quad \frac{8}{2}$	$\frac{4}{2} \quad 6 \div 2$	$6 \div 2 \quad 2 \times 3$
$2 \cdot 3 \quad \frac{10}{2}$	$3 \cdot 1 \quad \frac{6}{3}$	$3 \times 1 \quad 9 \div 3$
$4 \cdot 1 \quad \frac{4}{1}$	$8 \div 4 \quad \frac{8}{2}$	$\frac{12}{4} \quad \frac{4}{1}$
$\frac{5}{5} \quad \frac{1}{1}$	$\frac{10}{5} \quad 3 \cdot 2$	$15 \div 5 \quad 3 \cdot 2$
$2 \div 2 \quad 2 \cdot 2$	$12 \div 6 \quad 2 \cdot 1$	$\frac{18}{6} \quad 1 \cdot 3$
$4 \div 2 \quad 4 \times 2$	$\frac{14}{7} \quad 1 \times 7$	$\frac{21}{7} \quad 2 \cdot 3$

$\frac{2}{1} \quad \frac{4}{2} \quad \frac{6}{3}$	$\frac{8}{4} \quad \frac{10}{5} \quad \frac{12}{6}$
$\frac{3}{1} \quad \frac{6}{2} \quad \frac{9}{3}$	$\frac{12}{4} \quad \frac{15}{5} \quad \frac{18}{6}$
$\frac{12}{1} \quad \frac{12}{2} \quad \frac{12}{3}$	$\frac{12}{4} \quad \frac{12}{6} \quad \frac{12}{12}$

$2 \cdot 4 \quad 3 \cdot 3 \quad 4 \cdot 2$	$1 \cdot 5 \quad 2 \cdot 4 \quad 3 \cdot 3$
$3 \cdot 5 \quad 4 \cdot 4 \quad 5 \cdot 3$	$2 \cdot 6 \quad 3 \cdot 5 \quad 4 \cdot 4$
$4 \cdot 6 \quad 5 \cdot 5 \quad 6 \cdot 4$	$3 \cdot 7 \quad 4 \cdot 6 \quad 5 \cdot 5$

IF the white block is a picture suggesting "one,"

THEN yellow suggests "two,"
and blue suggests "four."

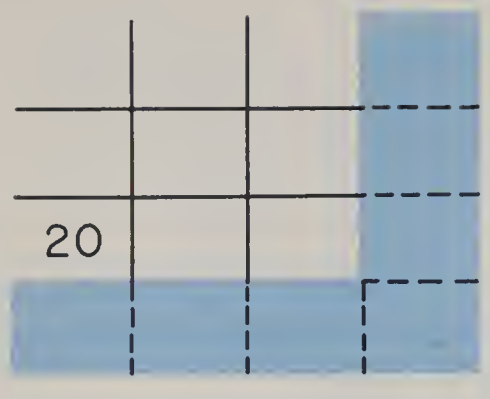


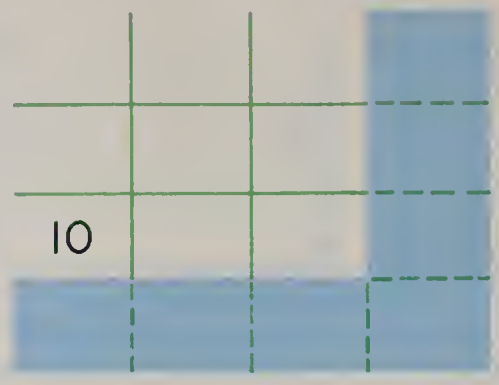
Fill in the chart below. In each example you are given a starting point.

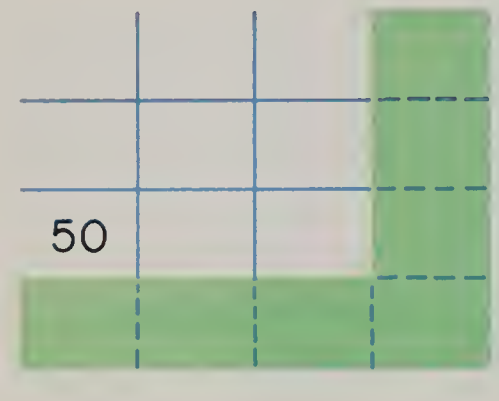
	W + Y	W + B	Y + O	B + Y
j	1			
k		1		
l			1	
m				1

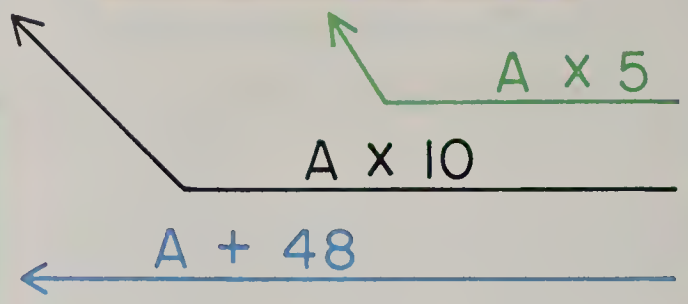
	W + Y + B	Y + B + O	W + B + O
n	1		
o		1	
p			1

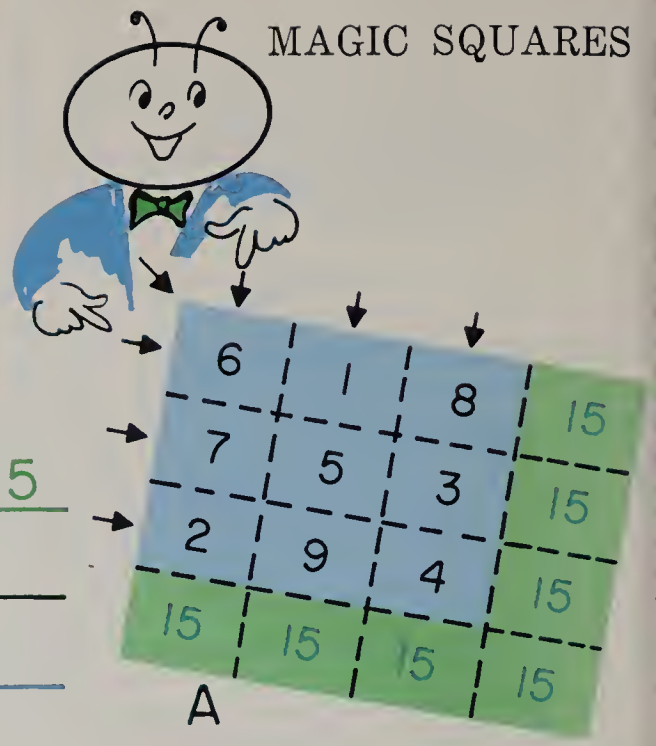
	White	Yellow	Blue	Orange
a		2		
b		6		
c	$\frac{1}{6}$			
d	$\frac{1}{3}$			
e			$\frac{4}{5}$	
f	$\frac{1}{2}$			
g		$\frac{1}{2}$		
h			$\frac{1}{2}$	
i				$\frac{1}{2}$











□ → _____ 2 x △ → _____ ○ + (3 x □) → _____

a. □ − △ → _____

b. △ x □ → _____

c. 10 x □ → _____

d. △ ÷ □ → _____

e. ○ + △ → _____

f. ○ + ○ → _____

g. ○ + ○ + △ → _____

h. □ − (2 x △) → _____

i. ○ + △ + ○ → _____

j. △ + ○ + □ → _____

k. △ x □ x ○ → _____

l. ○ + □ + □ → _____

m. 3 x ○ − ○ → _____

n. ○ + ○ + □ → _____

o. □ + △ − △ → _____

p. ○ x ○ + □ → _____

q. ○ + △ − △ → _____

r. (□ x △) ÷ ○ → _____

Complete the following magic boxes

	3		
		9	
	27	12	
	45		

83			
	59		
51		35	

	37		
	59	15	
26			
			147

		153	576
166			
	140	205	

	$\frac{1}{3}$		
$1\frac{2}{3}$	5		
		1	
			13

	1		
97	193		
	385	49	

A

$\frac{1}{4}$	$\frac{1}{4}$	
$\frac{1}{4}$	$\frac{1}{4}$	

B

$\frac{3}{4}$		
	$\frac{1}{4}$	
		1

C

$\frac{1}{2}$	$\frac{1}{4}$	
	$\frac{1}{4}$	
$\frac{1}{2}$		

D

$\frac{1}{8}$	$\frac{1}{8}$	
$\frac{3}{8}$	$\frac{3}{8}$	

E

$\frac{1}{8}$	$\frac{1}{4}$	
$\frac{3}{8}$	$\frac{1}{4}$	

F

$\frac{1}{8}$	$\frac{1}{8}$	
$\frac{1}{4}$	$\frac{1}{2}$	

G

$\frac{1}{2}$	$\frac{1}{4}$	
		$\frac{1}{4}$
	$\frac{1}{4}$	

H

		$\frac{3}{8}$
	$\frac{1}{8}$	
	$\frac{1}{4}$	1

I

	$\frac{1}{8}$	$\frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{2}$	

J

$\frac{7}{8}$		
	0	
	$\frac{1}{8}$	1

K

$\frac{5}{8}$		
	$\frac{3}{8}$	
		1

L

		$\frac{1}{2}$
	$\frac{1}{4}$	$\frac{1}{2}$
$\frac{3}{8}$		

$\frac{30}{2} = \frac{+}{\quad} = + =$

$\frac{30}{2} = \frac{+}{\quad} = + =$

$\frac{30}{2} = \frac{+\ +}{\quad} = + + =$

$\frac{30}{\ } = \frac{+\ +}{2} = + + =$

$\frac{30}{3} = \frac{+\ +}{\quad} = + + = 10$

$\frac{\ }{3} = \frac{+\ -}{\quad} = + - = 10$

$\frac{30}{\ } = \frac{-\ +}{\quad} = - + = 6$

$\frac{48}{2} = \frac{+}{2} = + =$

$\frac{48}{3} = \frac{+}{3} = + =$

$\frac{48}{4} = \frac{+\ +}{4} = + + =$

$\frac{48}{6} = \frac{+\ +}{6} = + + =$

$\frac{48}{8} = \frac{+\ -}{8} = + - =$

$\frac{48}{12} = \frac{-\ +}{12} = - + =$

$\frac{48}{16} = \frac{-\ -}{16} = - - =$

$42 \div 2 = (\ + \ ) \div 2 = (\ \div 2) + (\ \div 2) = \ + \ =$

$42 \div 2 = (\ + \ + \ ) \div 2 = (\ \div 2) + (\ \div 2) + (\ \div 2) = \ + \ + \ =$

$42 \div 3 = (\ + \ + \ ) \div 3 = (\ \div 3) + (\ \div 3) + (\ \div 3) = \ + \ + \ =$

$42 \div 7 = (\ + \ - \ ) \div 7 = (\ \div 7) + (\ \div 7) - (\ \div 7) = \ + \ - \ =$

$42 \div 6 = (\ - \ + \ ) \div 6 = (\ \div 6) - (\ \div 6) + (\ \div 6) = \ - \ + \ =$

$3\overline{)24} = 3\overline{)\ +\ } = 3\overline{)\ +\ +\ } = 3\overline{)\ +\ +\ +\ } = ---$

$4\overline{)36} = 4\overline{)\ +\ } = 4\overline{)\ +\ -\ } = 4\overline{)\ +\ +\ -\ } = ---$

$5\overline{)40} = 5\overline{)\ +\ } = 5\overline{)\ -\ +\ } = 5\overline{)\ +\ -\ +\ } = ---$

$6\overline{)60} = 6\overline{)\ +\ } = 6\overline{)\ -\ -\ } = 6\overline{)\ -\ +\ +\ } = ---$

ODD-EVEN Games

Rules for this page:

In the blue blocks ■ write only odd numbers.

In the green blocks ■ write only even numbers.

(Use 1 or 0 only when necessary.)

a. $\frac{\square}{3} = \square$

b. $\frac{\square}{7} = \square$

c. $\frac{12}{\square} = \square$

d. $\frac{24}{\square} = \square$

e. $\frac{18}{\square} = \square$

f. $\frac{42}{\square} = \square$

g. $\frac{28}{\square} = \square$

h. $\frac{\square}{\square} = 10$

i. $\frac{\square}{\square} = 17$

j $\frac{\square}{11} = \square$

k. $\frac{26}{\square} = \square$

l. $\frac{44}{\square} = \square$

m. $\frac{45}{\square} = \square$

n. $\frac{40}{\square} = \square$

o. $\frac{51}{\square} = \square$

p. $\frac{19}{\square} = \square$

q. $\frac{38}{\square} = \square$

r. $\frac{\square}{\square} = 31$

s. $\frac{\square}{\square} = 27$

t. $\frac{\square}{\square} = 22$

u. $\frac{\square}{\square} = 14$

v. $\frac{\square}{17} = \square$

A. $\frac{12}{3} = \frac{\square + \square}{3} = \frac{\square}{3} + \frac{\square}{3} = \square + \square = $ ____

B. $\frac{\square}{5} = \frac{\square + \square}{5} = \frac{\square}{5} + \frac{\square}{5} = \square + \square = $ ____

C. $\frac{15}{3} = \frac{\square + \square}{3} = \frac{\square}{3} + \frac{\square}{3} = \square + \square = $ ____

D. $\frac{18}{6} = \frac{\square - \square}{6} = \frac{\square}{6} - \frac{\square}{6} = \square - \square = $ ____

E. $\frac{40}{5} = \frac{\square - \square}{5} = \frac{\square}{} - \frac{\square}{} = \square - \square = $ ____

F. $\frac{16}{2} = \frac{\square - \square}{2} = \frac{\square}{2} - \frac{\square}{2} = \square - \square = $ ____

G. $\frac{21}{7} = \frac{\square}{7} + \frac{\square}{7} = $ ____

H. $\frac{24}{3} = \frac{\square}{3} + \frac{\square}{3} = $ ____

I. $\frac{10}{2} = \frac{\square}{2} - \frac{\square}{2} = $ ____

J. $\frac{30}{3} = \frac{\square}{3} - \frac{\square}{3} = $ ____

K. $\frac{42}{6} = \frac{\square}{6} + \frac{\square}{6} = $ ____

L. $\frac{35}{7} = \frac{\square}{7} - \frac{\square}{7} = $ ____

More ODD–EVEN Games

In the blue blocks ■ write only odd numbers.

In the green blocks ■ write only even numbers.

(Use 1 or 0 only when necessary.)

I $24 \div 3 = (\square + \square) \div \square = \square \div \square + \square \div \square = \square + \square = \square$

II $25 \div \square = (\square + \square) \div \square = \square \div \square + \square \div \square = \square + \square = \square$

III $28 \div \square = (\square + \square) \div \square = \square \div \square + \square \div \square = \square + \square = \square$

IV $12 \div \square = (\square - 8) \div \square = \square \div 4 - \square \div \square = \square - \square = \square$

V $30 \div \square = (40 - \square) \div \square = \square \div \square - \square \div \square = 20 - \square = \square$

VI $\square \div \square = (\square - \square) \div 7 = 56 \div \square - \square \div \square = \square - \square = 6$

VII $\square\overline{)36} = \square\overline{)24 + \square} = \square\overline{)\square - 12} = \square\overline{)\square + 20 - \square} = 9$

VIII $\square\overline{)45} = \square\overline{)30 + \square} = \square\overline{)60 - \square} = \square\overline{)45 - \square + \square} = \square$

IX $\square\overline{)\square} = \square\overline{)\square + 6} = \square\overline{)75 - 20} = \square\overline{)50 + \square - 40} = 11$

X $\square\overline{)\square} = \square\overline{)\square + 12} = \square\overline{)54 + 18} = \square\overline{)\square + \square + \square} = \square$

XI $7\overline{)91} = \square\overline{)\square + \square} = \square\overline{)\square + \square} = \square\overline{)\square - 49 + 63} = \square$

XII $3\overline{)102} = \square\overline{)\square \div \square} = \square\overline{)\square - \square} = \square\overline{)\square + 63 - \square} = \square$

XIII $8\overline{)120} = \square\overline{)\square + \square} = \square\overline{)\square - \square} = \square\overline{)\square - 16 + 64} = \square$

Another **ODD-EVEN** Game

Rule: In the blue section use odd numbers whenever you can. Use 1 only when necessary.

In the green section use all the even numbers you can. Use 0 only when necessary.

a. $\frac{12}{} = \frac{+}{} = + = 4$

b. $\frac{32}{} = \frac{+}{} = + = 8$

c. $\frac{9}{)45} = \frac{+\ +}{)\ +\ +} = ---$

d. $2\overline{)58} = 2\overline{)\ +\ +} = ---$

e. $7\overline{)98} = 7\overline{)\ +\ +} = ---$

f. $3\overline{)69} = 3\overline{)\ -\ +} = ---$

l. $5\overline{)65} = 5\overline{)\ +} = 5\overline{)\ -} = 5\overline{)\ +} = 5\overline{)\ +\ +} = ---$

m. $6\overline{)84} = 6\overline{)30+} = 6\overline{)\ -} = 6\overline{)\ +} = 6\overline{)\ +\ -} = ---$

n. $3\overline{)72} = 3\overline{)\ +} = 3\overline{)\ +} = 3\overline{)\ -} = 3\overline{)\ -\ -} = ---$

o. $4\overline{)96} = 4\overline{)\ +} = 4\overline{)\ -} = 4\overline{)\ -} = 4\overline{)\ -\ -} = ---$

g. $\frac{32}{} = \frac{+}{} = + = 8$

h. $\frac{45}{} = \frac{+}{} = + = 9$

i. $2\overline{)58} = 2\overline{)\ +\ +} = ---$

j. $7\overline{)98} = 7\overline{)\ +\ +} = ---$

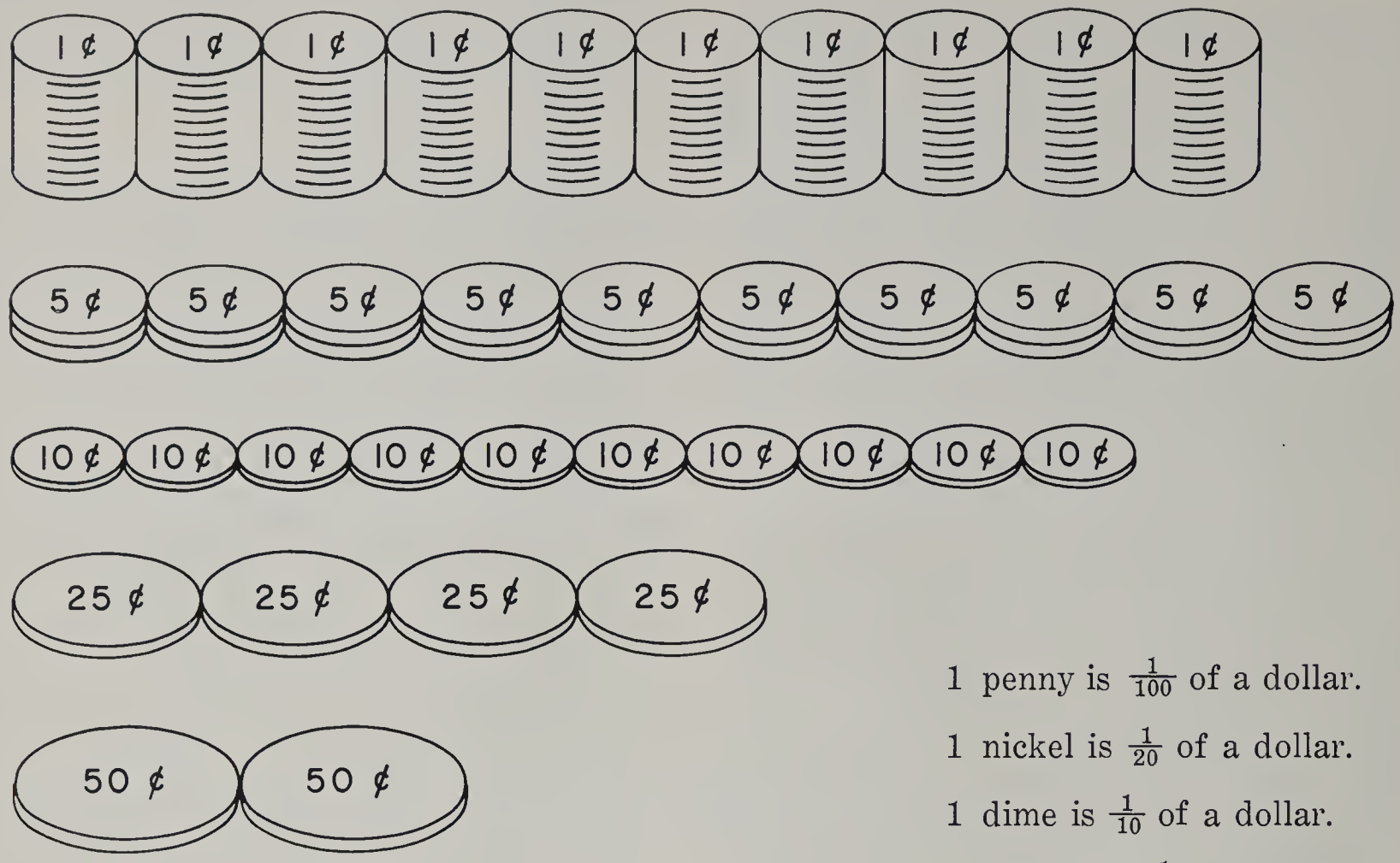
k. $3\overline{)69} = 3\overline{)\ -\ +} = ---$

Only rules: Don't use 0 or 1 unless necessary. Don't repeat more than you need to.

p. $2\overline{)218} = 2\overline{)\ +\ +} = 2\overline{)\ +\ +} = 2\overline{)\ +} = ---$

q. $3\overline{)324} = 3\overline{)\ +\ +} = 3\overline{)\ +\ +} = 3\overline{)\ +} = ---$

r. $5\overline{)525} = 5\overline{)\ +\ +} = 5\overline{)\ +\ +} = 5\overline{)\ +} = ---$



1 penny is $\frac{1}{100}$ of a dollar.

1 nickel is $\frac{1}{20}$ of a dollar.

1 dime is $\frac{1}{10}$ of a dollar.

1 quarter is $\frac{1}{4}$ of a dollar.

1 half is $\frac{1}{2}$ of a dollar.

Amount as fraction of a dollar	Fewest coins — written as fractions of a dollar
$\frac{7}{100}$	$\frac{1}{20} + \frac{1}{100} +$ ___
$\frac{35}{100}$	___ + ___ ___
$\frac{3}{4}$	___ ___ ___
$\frac{31}{100}$	___ ___ ___
$\frac{6}{10}$	___ ___ ___
$\frac{5}{20}$	___ ___ ___
$\frac{16}{100}$	___ ___ ___
$\frac{7}{10}$	___ ___ ___

$\frac{8}{20}$	___ ___ ___
$\frac{1}{5}$	___ ___ ___
$\frac{3}{5}$	___ ___ ___
$\frac{56}{100}$	___ ___ ___
$\frac{13}{50}$	___ ___ ___
$\frac{4}{25}$	___ ___ ___
$\frac{13}{20}$	___ ___ ___

An Agreement: For just this one page, we are going to forget about all money except silver dollars, dimes, and pennies.

Words	Three kinds of shorthand A	B	C	In C, write twice as much	half as much
one dollar	$ 1	100 ¢	$ 1.00	$ 2.00	$ .50
one dime	$ $\frac{1}{10}$		$ .10		
one penny	$ $\frac{1}{100}$		$ .01		$ .00$\frac{1}{2}$
one and one-half dollars					
fifteen dimes					
twenty-four cents					
two dollars, forty-eight cents					
one hundred eighty-four cents					
one dollar and a quarter					
three dollars seventy-five cents					
eight hundred twenty-two cents					

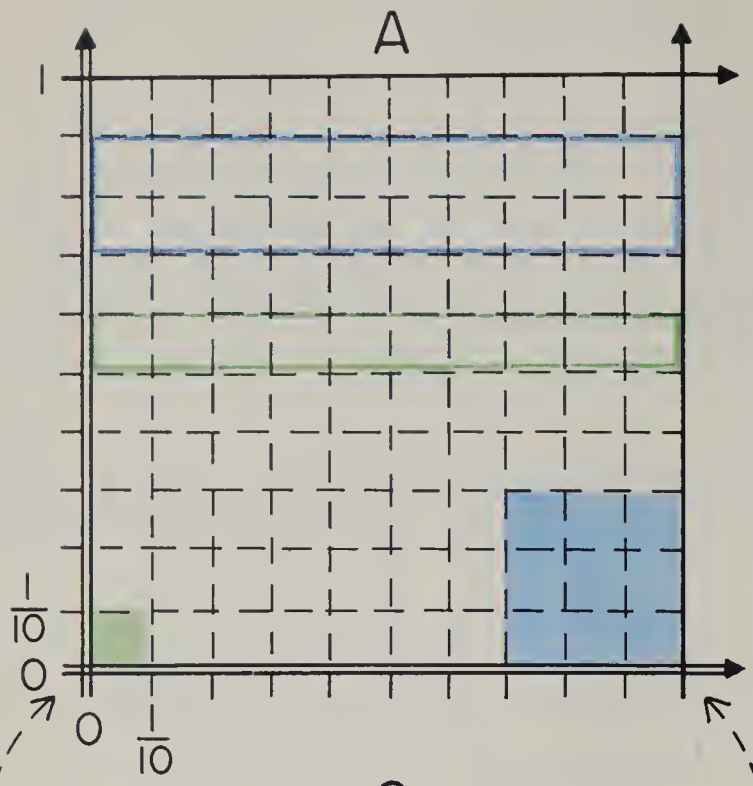
Summer Refreshment Stand

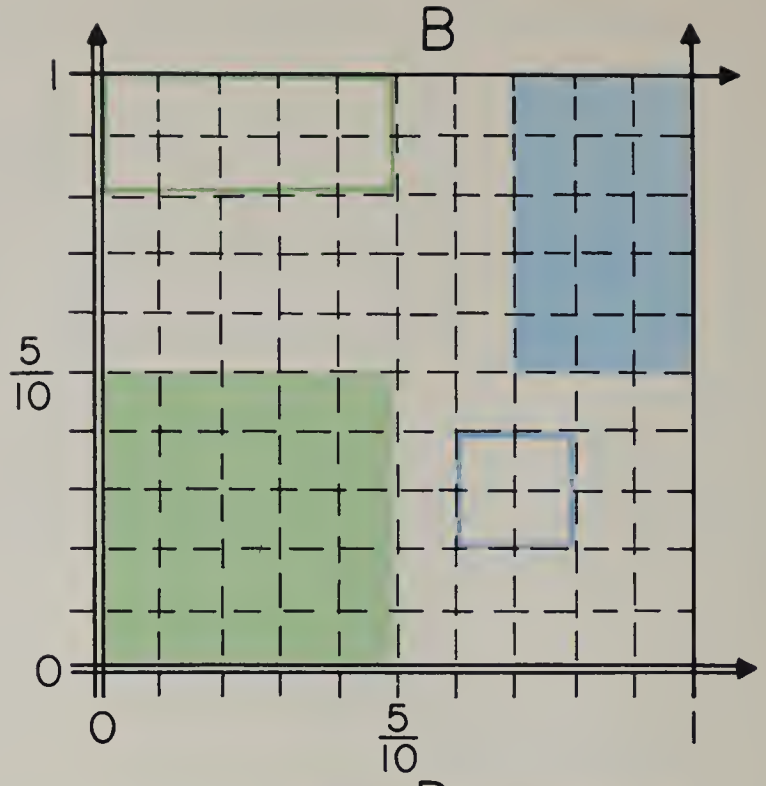
Mary, Alice, and Jane operated a refreshment stand one summer.

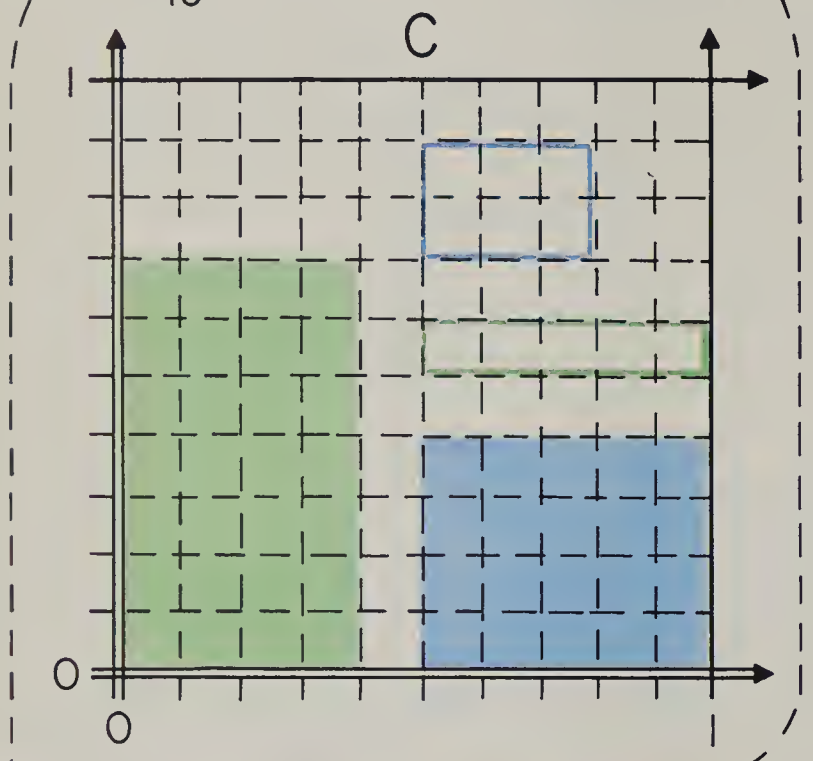
Each day, they paid for everything they sold. The money they had left was profit. Each day, they divided the profits evenly among themselves.

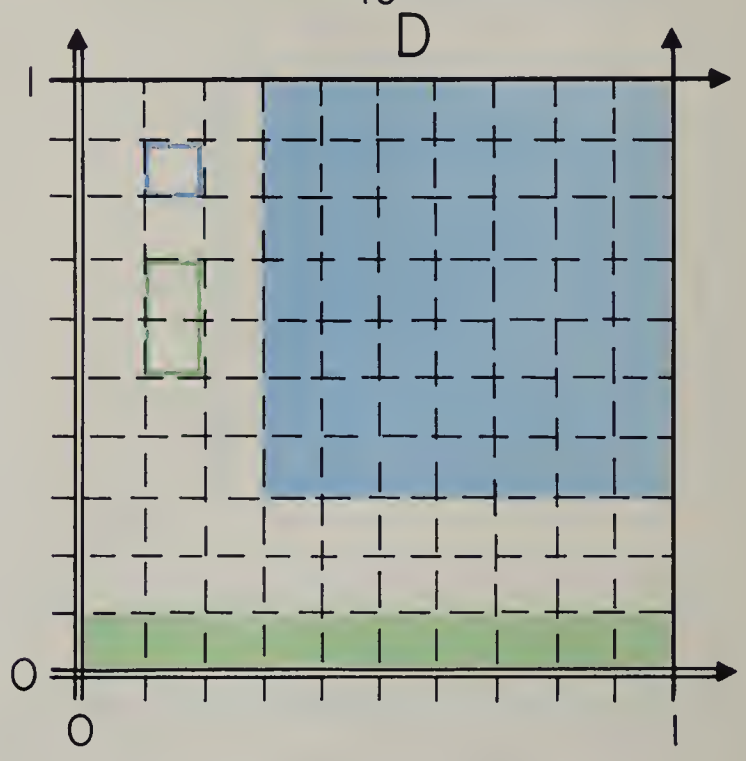
Below are partial records of one week's activity. Can you complete the record?

	Profit in numbers of			Profit in number of fewest coins			Each child's share in number of			Pennies left over
	Dollars	Dimes	Pennies	Dollars	Dimes	Pennies	Dollars	Dimes	Pennies	
Mon.	1	7	22							
Tues.	2	11	16							
Wed.	1	18	36							
Thurs.	0	21	41							
Fri.	1	13	88							
Sat.	0	27	58							
Totals										









A → $\frac{1}{10} \times \frac{}{10} = \frac{}{100}$

A → $\frac{10}{10} \times \frac{1}{10} = \frac{}{100}$

A → $\frac{10}{10} \times \frac{2}{10} = \frac{}{100}$

B → $\frac{}{} \times \frac{}{} = \frac{}{100}$

B → $\frac{}{} \times \frac{}{} = \frac{}{}$

B → $\frac{}{} \times \frac{}{} = \frac{}{100}$

C → $\frac{}{} \times \frac{}{} = \frac{}{}$

C → $\frac{}{} \times \frac{}{} = \frac{}{100}$

D → $\frac{10}{10} \times \frac{}{} = \frac{}{100}$

D → $\frac{}{} \times \frac{}{} = \frac{}{}$

D → $\frac{}{} \times \frac{}{} = \frac{}{}$

A → $\frac{3}{10} \times \frac{3}{10} = \frac{}{100}$

B → $\frac{}{} \times \frac{}{} = \frac{}{100}$

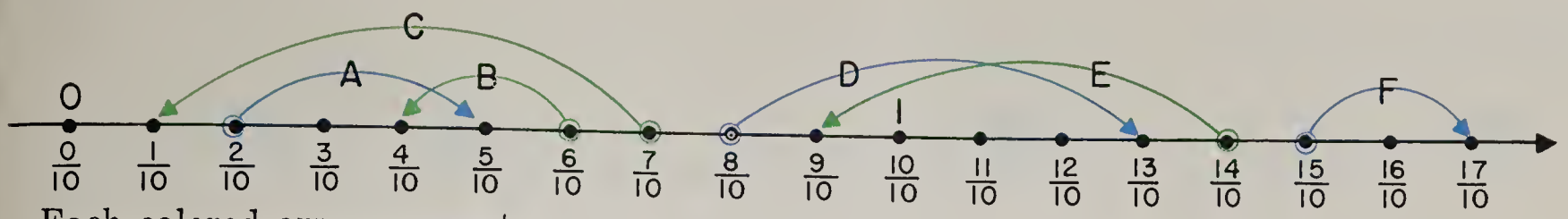
B → $\frac{}{} \times \frac{}{} = \frac{}{100}$

C → $\frac{}{} \times \frac{}{} = \frac{}{}$

C → $\frac{}{} \times \frac{}{} = \frac{}{}$

D → $\frac{}{} \times \frac{}{} = \frac{}{}$

A + D + B + B → $\frac{}{} + \frac{}{} + \frac{}{} + \frac{}{} = \frac{}{}$



Each colored arrow suggests:

A. $\frac{2}{10} + \frac{\quad}{10} = \frac{5}{10}$ B. $\frac{\quad}{10} - \frac{\quad}{10} = \frac{\quad}{10}$ C. $\frac{\quad} - \frac{\quad} = \frac{\quad}{10}$

D. $\frac{\quad} + \frac{\quad} = \frac{\quad}{10}$ E. $\frac{\quad} - \frac{\quad} = \frac{\quad}$ F. $\frac{\quad} + \frac{\quad} = \frac{\quad}$

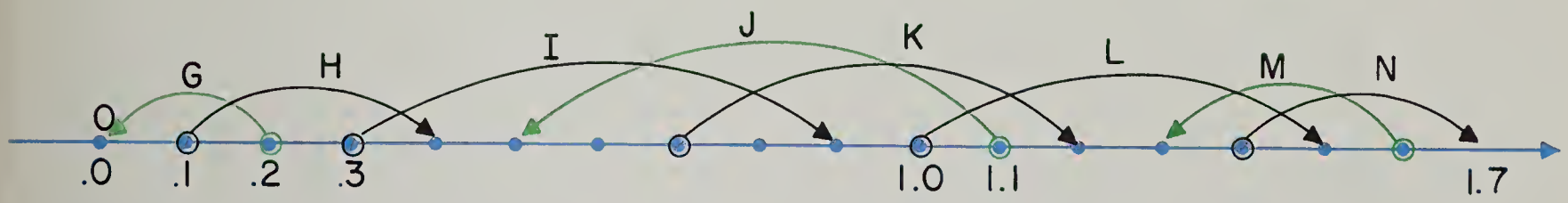
More shorthand: We can save much time by agreeing

that $\frac{1}{10}$ can be written .1 and

that $\frac{7}{10}$ can be written .7 and

that $\frac{17}{10}$ can be written 1.7

Finish labeling the numberline below in this new shorthand.



These arrows suggest:

G. .2 − .2 = .0 H. .1 + ___ = ___ I. ___ + ___ = ___ J. ___ − ___ = ___

K. ___ + ___ = ___ L. ___ + ___ = ___ M. ___ − ___ = ___ N. ___ + ___ = ___

Complete the records (unless the block is colored green).

Instructions below apply to number written in blue box	.1	.9	1.2	.0	.6	1.0	1.8	2.1	3.0
Double each number	.2	1.8	2.4						
Add two tenths to each	.3	1.1							
Multiply each by three	.3								
Add ten tenths to each	1.1								
Subtract one tenth from each	.0								
Add twenty tenths to each	2.1								
Divide each by two									
Divide each by three									
Double each and then add three tenths	5								

Find the missing factor.

$\square \times \frac{1}{10} = \frac{3}{10}$ or $\square \times .1 = .3$ or $.3 \div .1 = \square$ or $\square\overline{).3}$ (divisor .1)

$.6\overline{)\square}$ (dividend 1.2) or $\square\overline{)1.2}$ (divisor .6) or $\square \times .6 = 1.2$ or $\square \times \frac{6}{10} = \frac{12}{10}$

$5 \times \square = \frac{25}{10}$ or $5 \times \square = 2.5$ or $5\overline{)\square}$ (dividend 2.5) or $\square\overline{)2.5}$ (divisor 5)

$\_\_ \times .8 = 5.6$ or $\_\_ \times (.3 + .5) = (\_\_ \times .3) + (\_\_ \times .5) = \_\_ + \_\_ = 5.6$

$11 \times \_\_ = \frac{55}{10}$ or $(2 + 9) \times \_\_ = (2 \times \_\_) + (9 \times \_\_) = \_\_ + \_\_ = 5.5$

Name pairs of missing factors (don't repeat unless necessary).

$\_\_ \times \_\_ = 1.2$	$\_\_ \times \_\_ = 1.2$	$\_\_ \times \_\_ = 1.2$	$\_\_ \times \_\_ = 1.2$
$\_\_ \times \_\_ = 3.0$	$\_\_ \times \_\_ = 3.0$	$\_\_ \times \_\_ = 3.0$	$\_\_ \times \_\_ = 3.0$
$\_\_ \times \_\_ = 3.0$	$\_\_ \times \_\_ = 3.0$	$\_\_ \times \_\_ = 3.0$	$\_\_ \times \_\_ = 3.0$
$\_\_ \times \_\_ = 4.8$	$\_\_ \times \_\_ = 4.8$	$\_\_ \times \_\_ = 4.8$	$\_\_ \times \_\_ = 4.8$
$\_\_ \times \_\_ = 4.8$	$\_\_ \times \_\_ = 4.8$	$\_\_ \times \_\_ = 4.8$	$\_\_ \times \_\_ = 4.8$

$\square\overline{)3.6}$ (divisor) $\quad \square\overline{)3.6}$ (dividend) $\quad \square\overline{)3.6}$ (quotient) $\quad \square\overline{)3.6}$ (dividend) $\quad \square\overline{)3.6}$ (divisor)

$\square\overline{)3.2 + 8}$ (dividend) $\quad \square\overline{)1.2 + 2.0 + 8}$ (dividend) $\quad \square\overline{)2.4 + .8 + 8}$ (dividend)

Use the new shorthand.

$2 = \frac{\square}{10} = \square$ $\quad \frac{1}{2} = \frac{\square}{10} = \square$ $\quad \frac{1}{5} = \frac{\square}{10} = \square$ $\quad 1\frac{1}{5} = \frac{\square}{10} = \square$

$9 = \frac{\square}{10} = \square$ $\quad \frac{3}{2} = \frac{\square}{10} = \square$ $\quad \frac{2}{5} = \frac{\square}{10} = \square$ $\quad 2\frac{4}{5} = \frac{\square}{10} = \square$

$10 = \frac{\square}{10} = \square$ $\quad 2\frac{1}{2} = \frac{\square}{10} = \square$ $\quad \frac{5}{5} = \frac{\square}{10} = \square$ $\quad 1 = \frac{\square}{10} = \square$

How far down the page can you go with ease and without help?

$2\overline{)322} = 2\overline{)200+100+22} = 2\overline{)160+160+} = 2\overline{)300+} = $ ----

$3\overline{)402} = 3\overline{)\quad+99+3} = 3\overline{)\quad+\quad+} = 3\overline{)\quad+} = $ ----

$5\overline{)825} = 5\overline{)500+\quad+} = 5\overline{)\quad+\quad+} = 5\overline{)\quad+} = $ ----

$12\overline{)156} = 12\overline{)\quad+\quad+} = 12\overline{)\quad+\quad+} = 12\overline{)\quad+} = $ ----

$20\overline{)480} = 20\overline{)200+200+} = 20\overline{)\quad+\quad+} = 20\overline{)\quad+} = $ ----

$22\overline{)246} = 22\overline{)\quad+\quad+} = 22\overline{)\quad+\quad+} = 22\overline{)\quad+} = $ ----

$25\overline{)675} = 25\overline{)\quad+\quad+} = 25\overline{)\quad+\quad+} = 25\overline{)\quad+} = $ ----

$28\overline{)700} = 28\overline{)\quad+\quad+} = 28\overline{)\quad+\quad+} = 28\overline{)\quad+} = $ ----

$23\overline{)851} = 23\overline{)\quad+\quad+} = 23\overline{)\quad+\quad+} \quad 23\overline{)\quad+} = $ ----

$17\overline{)952} = 17\overline{)\quad+\quad+} = 17\overline{)\quad+\quad+} = 17\overline{)\quad+} = $ ----

$31\overline{)992} = 31\overline{)\quad+\quad+} = 31\overline{)\quad+\quad+} = 31\overline{)\quad+} = $ ----

$53\overline{)954} = 53\overline{)\quad+\quad+} = 53\overline{)\quad+\quad+} = 53\overline{)\quad+} = $ ----

$27\overline{)756} = 27\overline{)\quad+\quad+} = 27\overline{)\quad+\quad+} = 27\overline{)\quad+} = $ ----

Another Division Game: Rule: In the blue blocks ■ write only multiples of 5
In the green blocks ■ write only numbers less than 5

a. $5\overline{)55} = 5\overline{)\square + \square} \;(+4) = 5\overline{)\square + \square + \square} = 5\overline{)\square + \square} \;(+6) = \_\_\_\_$

b. $4\overline{)36} = 4\overline{)\square + \_\_\_\_} \;(+\square) = 4\overline{)\_\_\_\_ + \_\_\_\_ + \_\_\_\_} = 4\overline{)\square - 24} \;(-\square) = \_\_\_\_$

c. $3\overline{)48} = 3\overline{)15 + \_\_\_\_} \;(15 + \square) = \_\_\_\_$

d. $6\overline{)42} = 6\overline{)\_\_\_\_ + \square} \;(\square + \square) = \_\_\_\_$

e. $7\overline{)63} = 7\overline{)\square + \_\_\_\_} \;(\square + \square) = \_\_\_\_$

f. $8\overline{)216} = 8\overline{)\square + \_\_\_\_} \;(\square + \square) = \_\_\_\_$

g. $6\overline{)108} = 6\overline{)\_\_\_\_ + \square} \;(\square + \square) = \_\_\_\_$

h. $2\overline{)78} = 2\overline{)\square + \_\_\_\_} \;(\square + \square) = \_\_\_\_$

Another rule change: In the green blocks ■ write multiples of 10
In the blue blocks ■ write numbers less than 10

i. $2\overline{)28} = 2\overline{)\square + \_\_\_\_} \;(\square + \square) = \_\_\_\_$

j. $3\overline{)69} = 3\overline{)\_\_\_\_ + \square} \;(\square + \square) = \_\_\_\_$

k. $3\overline{)78} = 3\overline{)\square + \_\_\_\_} \;(\square + \square) = \_\_\_\_$

l. $4\overline{)100} = 4\overline{)\square + \_\_\_\_} \;(\square + \square) = \_\_\_\_$

m. $5\overline{)160} = 5\overline{)\square + \_\_\_\_} \;(\square + \square) = \_\_\_\_$

n. $4\overline{)148} = 4\overline{)\_\_\_\_ + \square} \;(\square + \square) = \_\_\_\_$

o. $3\overline{)108} = 3\overline{)\square + \_\_\_\_} \;(\square + \square) = \_\_\_\_$

p. $5\overline{)325} = 5\overline{)\square + \_\_\_\_} \;(\square + \square) = \_\_\_\_$

q. $7\overline{)147} = 7\overline{)\square + \_\_\_} \;(\square + \square) = \_\_\_\_$

r. $6\overline{)96} = 6\overline{)\square + \_\_\_} \;(\square + \square) = \_\_\_\_$

(continued from bottom, page 92)

In the green blocks ■ write multiples of 10

In the blue blocks ■ write numbers less than 10

A. $3\overline{)159} = 3\overline{)\square + \square} = \_\_\_\_\_$

B. $4\overline{)128} = 4\overline{)\square + \square} = \_\_\_\_\_$

C. $5\overline{)265} = 5\overline{)\square + \square} = \_\_\_\_\_$

D. $4\overline{)600} = 4\overline{)\square + \square} = \_\_\_\_\_$

E. $7\overline{)161} = 7\overline{)\square + \square} = \_\_\_\_\_$

F. $10\overline{)920} = 10\overline{)\square + \square} = \_\_\_\_\_$

G. $8\overline{)1608} = 8\overline{)\square + \square} = \_\_\_\_\_$

H. $11\overline{)495} = 11\overline{)\square + \square} = \_\_\_\_\_$

I. $3\overline{)84} = 3\overline{)\square - \square} = \_\_\_\_\_$

J. $5\overline{)130} = 5\overline{)\square - \square} = \_\_\_\_\_$

K. $6\overline{)96} = 6\overline{)\square - \square} = \_\_\_\_\_$

L. $7\overline{)119} = 7\overline{)\square - \square} = \_\_\_\_\_$

M. $6\overline{)198} = 6\overline{)\square - \square} = \_\_\_\_\_$

N. $9\overline{)414} = 9\overline{)\square - \square} = \_\_\_\_\_$

O. $11\overline{)253} = 11\overline{)\square - \square} = \_\_\_\_\_$

P. $12\overline{)156} = 12\overline{)\square - \square} = \_\_\_\_\_$

Make up and complete some of your own examples.

Q. $\overline{)\quad} = \overline{)\square + \square} = \_\_\_\_\_$

R. $\overline{)\quad} = \overline{)\square + \square} = \_\_\_\_\_$

S. $\overline{)\quad} = \overline{)\square + \square} = \_\_\_\_\_$

T. $\overline{)\quad} = \overline{)\square - \square} = \_\_\_\_\_$

U. $\overline{)\quad} = \overline{)\square - \square} = \_\_\_\_\_$

V. $\overline{)\quad} = \overline{)\square - \square} = \_\_\_\_\_$

Chain Reactions: Make each term in each row, half of the term on its left.

	First term									Last term
A	512									1
B		768								3
C	2,560									
D		1,792								
E			1,152							
F		2,816								
G			1,664							
H										
I										

If the first term is considered a product, and the last term one of its factors, what is the other factor?

A. $1\overline{)512}$ B. $3\overline{)\quad}$ C. $\overline{)\quad}$

D. $\overline{)\quad}$ E. $\overline{)\quad}$ F. $\overline{)\quad}$

G. $\overline{)\quad}$ H. $\overline{)\quad}$ I. $\overline{)\quad}$

Another Chain Reaction: Make each term in each row one-third of the term on its left.

J	19,683									
K		13,122								
L			8,748							
M			10,935							
N		45,927								
O	196,830									

More shorthand —

We have already agreed that

.1 is shorthand for $\frac{1}{10}$

.7 is shorthand for $\frac{7}{10}$

1.8 is shorthand for $\frac{18}{10}$, or $1\frac{8}{10}$, or $1\frac{4}{5}$

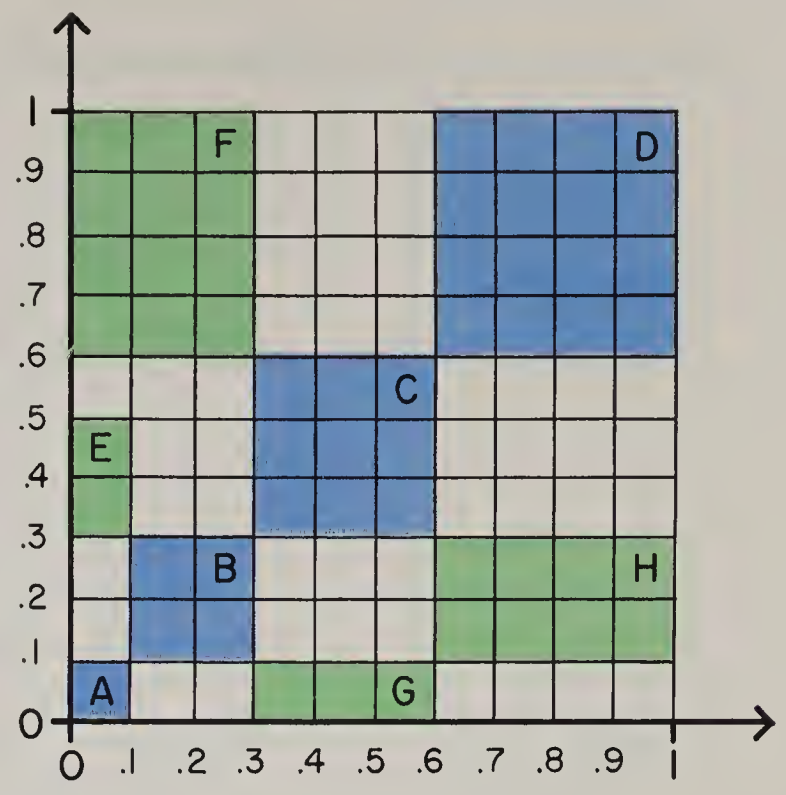
We extend that agreement —

.01 is shorthand for $\frac{1}{100}$

.06 is shorthand for $\frac{6}{100}$

.24 is shorthand for $\frac{24}{100}$

1.25 is shorthand for $\frac{125}{100}$



In the diagram (upper right), the colored blocks suggest the following:

A. $\frac{1}{10} \times \frac{1}{10} = \frac{1}{100}$ or .1 x .1 = .01

B. $\frac{2}{10}$ x — = — or .2 x ___ = ___

C. $\frac{3}{10}$ x — = — or .3 x ___ = ___

D. — x — = — or ___ x ___ = ___

E. — x — = — or ___ x ___ = ___

F. — x — = — or ___ x ___ = ___

G. — x — = — or ___ x ___ = ___

H. — x — = — or ___ x ___ = ___

$\frac{7}{10} = \frac{70}{100}$ or .7 = .70

$2 = \frac{20}{10} = \frac{200}{100}$ or 2 = 2.0 = 2.00

Combinations of the blocks lettered above may suggest the following:

A & B together: .01 + .04 = .05

B & C together: .04 + .09 =

C & D together: ___ + ___ = ___

D & E together: ___ + ___ = ___

E & F together: ___ + ___ = ___

F & G together: ___ + ___ = ___

G & H together: ___ + ___ = ___

A & C together: ___ + ___ = ___

C & E together: ___ + ___ = ___

E & G together: ___ + ___ = ___

B & D together: ___ + ___ = ___

D & F together: ___ + ___ = ___

F & H together: ___ + ___ = ___

B & F together: ___ + ___ = ___

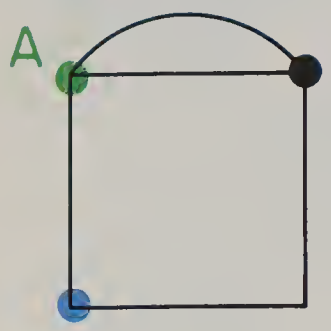
C & G together: ___ + ___ = ___

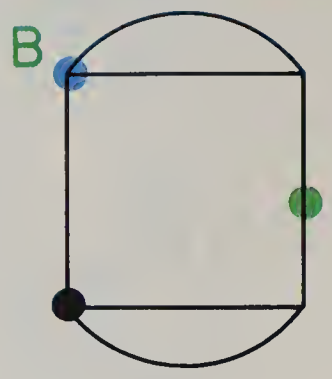
D & H together: ___ + ___ = ___

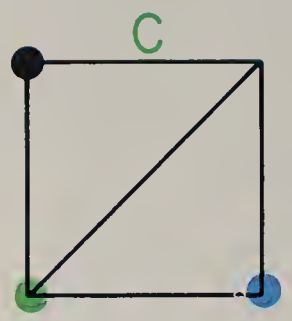
How many can you draw?

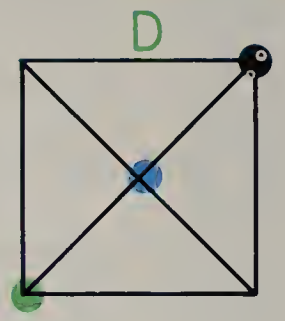
——— starting at the green dot
——— starting at the blue dot
——— starting at the black dot

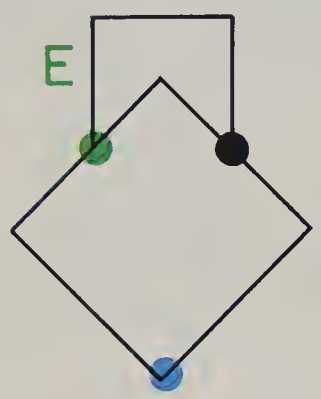
Rule: Without retracing and without lifting your pencil.

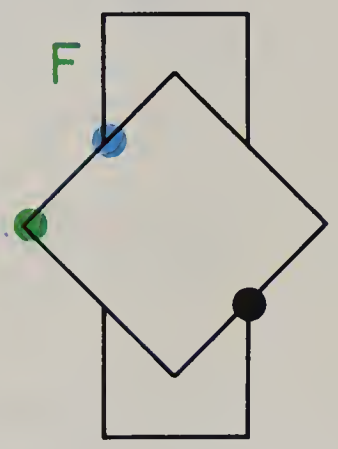


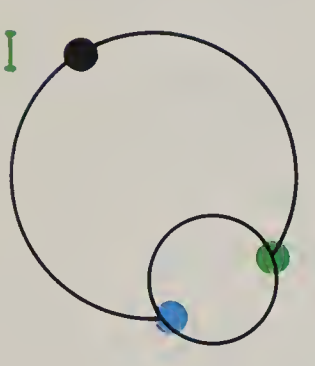


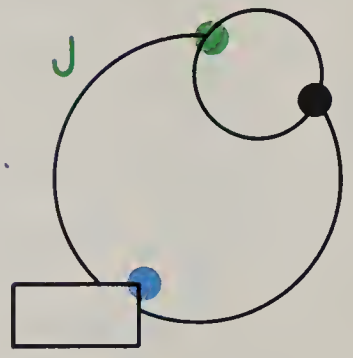


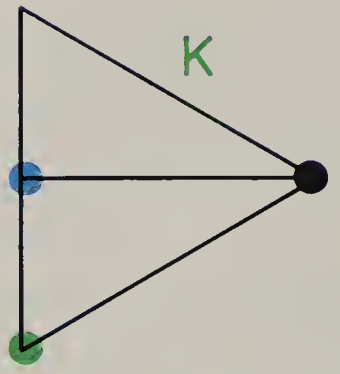


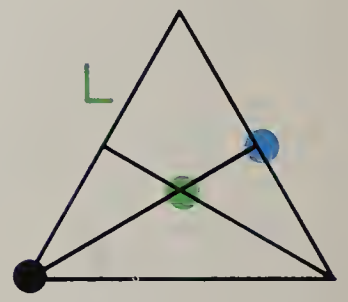


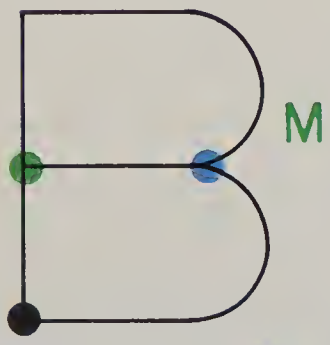


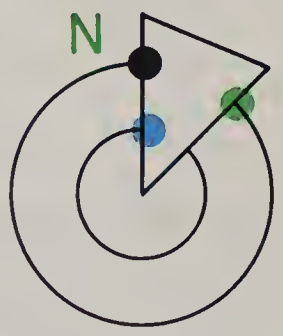


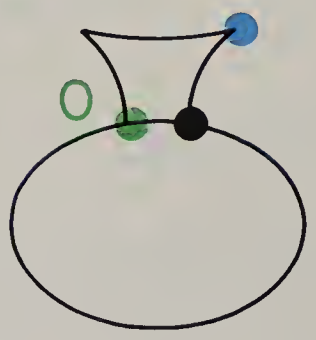


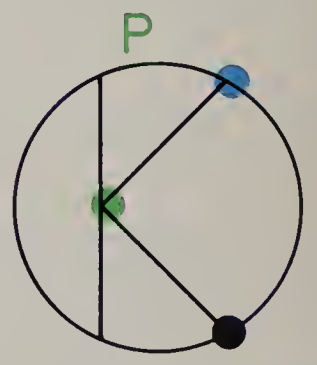








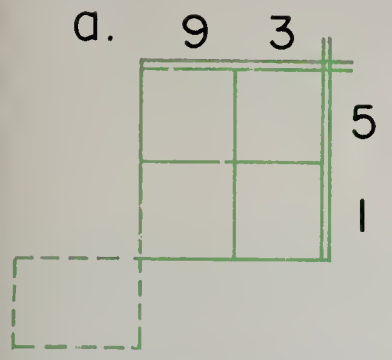




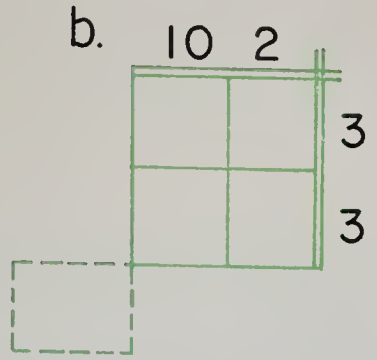
Can you draw it starting at:	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P
Green dot	Y	N														
Blue dot	N															
Black dot	Y															

Find the pattern in each row of Cross Number Puzzles.

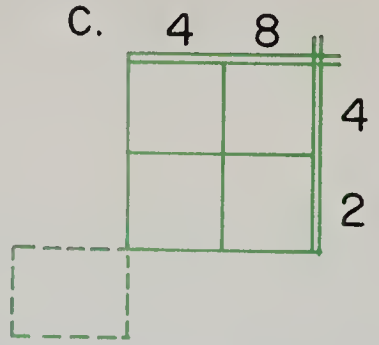
a.



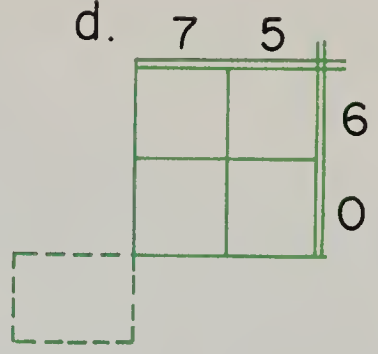
b.

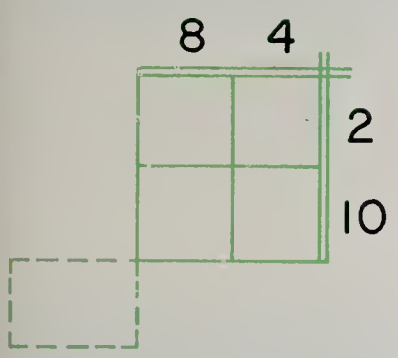


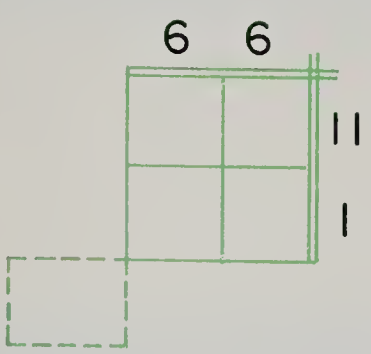
c.

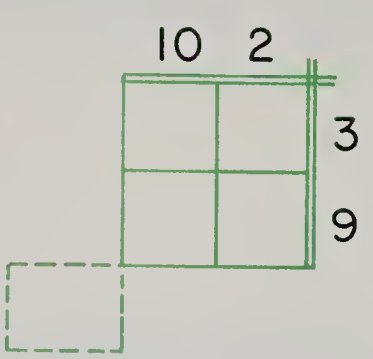


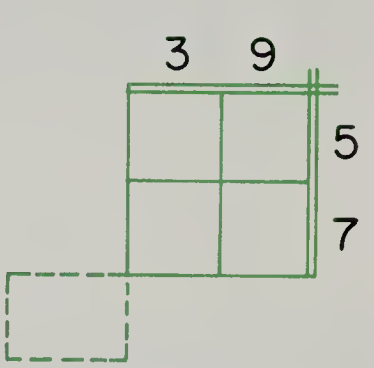
d.

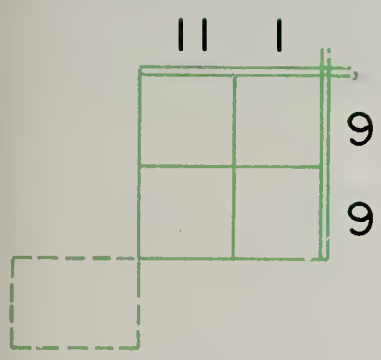


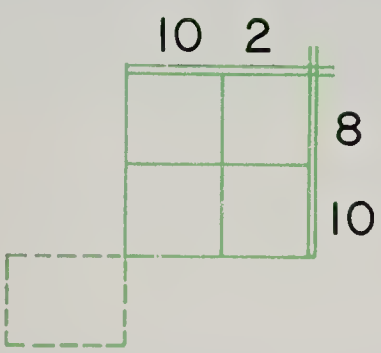


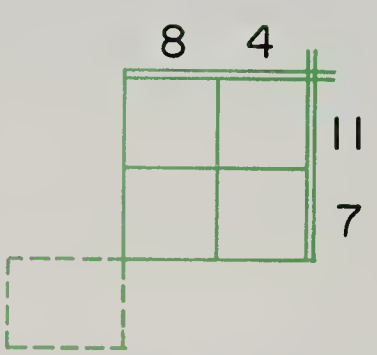


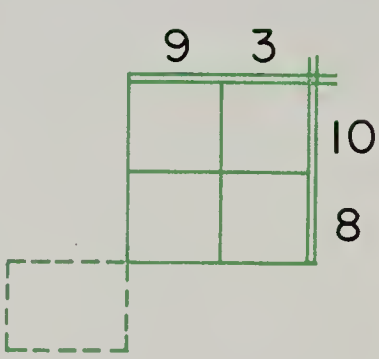




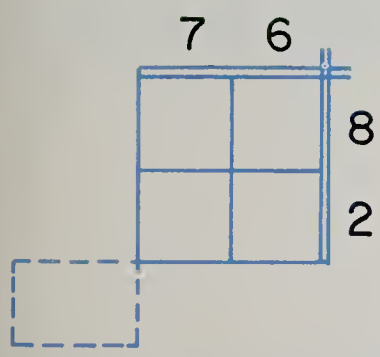


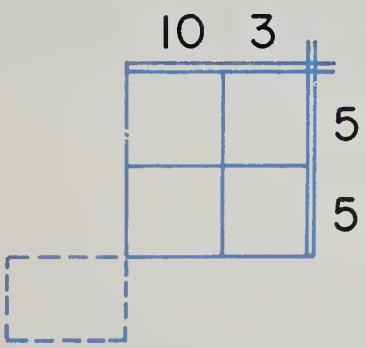


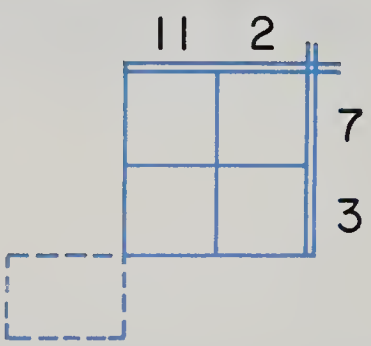


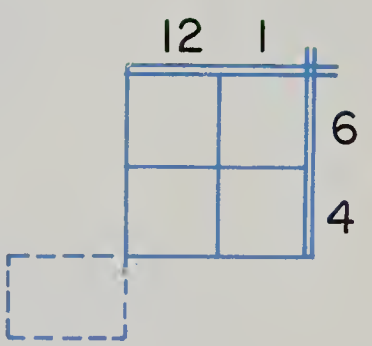


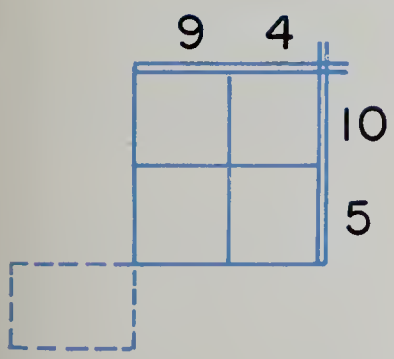
Then look for the same pattern below.

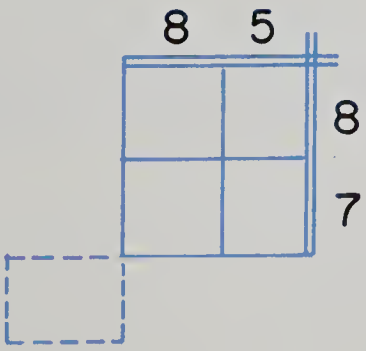


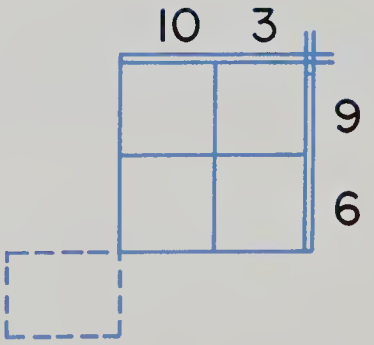


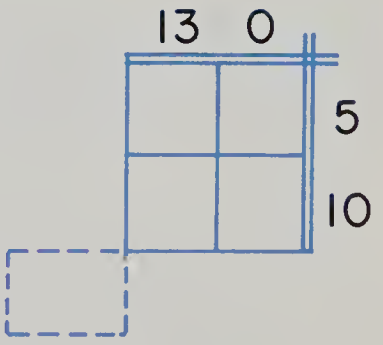


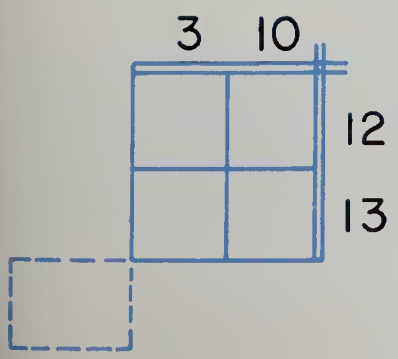


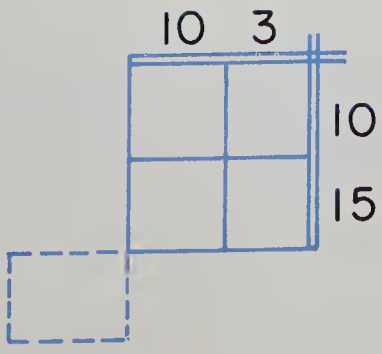


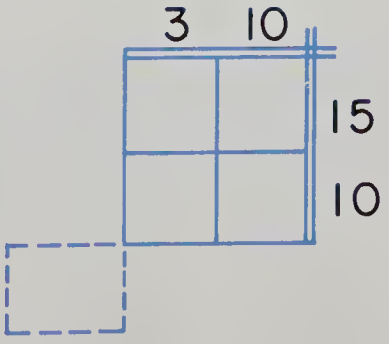


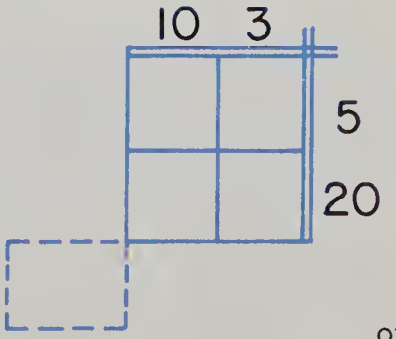






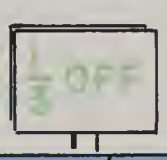






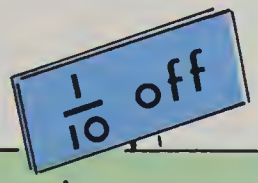
In preparing for a sale, the clerk in the store had to compute many new prices. Here are some of the records he made:

Regular price	$ 4.28	$ 3.00	$ 3.50	$ 10.82	$ 124.30
Amount of reduction					
Sale price					

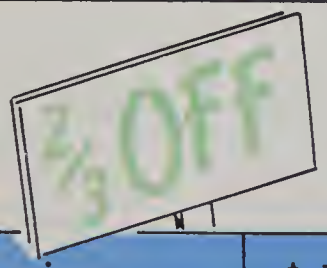


Regular price	$ 3.60	$ 4.50	$ 6.81	$ 123.75	$ 51.87
Amount of reduction					
Sale price					

Regular price	$ 8.60	$ 2.68	$ 9.28	$ 45.76	$ 150.00
Amount of reduction					
Sale price					



Regular price	$ 1.90	$ 35.00	$ 10.00	$ 100.00	$ 848.00
Amount of reduction					
Sale price					



Regular price	$ 7.50	$ .96	$ 12.87	$ 225.00	$ 29.34
Amount of reduction					
Sale price					

Find the Whole Number Factors (work in green area only if time permits)

If the number written at the top of the row is a whole number factor of the number on the right, mark Y. If not, mark N.

	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
24	Y	Y	Y	N	Y	N	Y	N	N						
60	Y	Y													
37															
210															
120															
720															
72															
64															
2,310															
256															
729															
216															
4,290															
121															
49															
3,003															
40,320															
5,040															
362,880															
555															
13,000															
1,000															
17,017															
123,456															

surprise!

THINK-OF-A-NUMBER!

Add 3 to it.

Double your result.

Take 4 away.

Divide your result by 2.

Take away the number you thought of first.

Ahah! Your result is 1.

Try this with your friends.

If they get a final result other than one, tell them that they must have made a mistake.

Will that surprise them!

You will learn to make up your own "think of a number" games.

Here is another "think of a number" game. Try four different examples.

	Examples: A	B	C	D
Think of a number	5	8	100	
Add 4	9	12		
Multiply by 3	27			
Take away the number you thought of first				
Divide by 2				
Take away 2				
Take away the number you thought of first				

Complete the columns below:

1 x 9 =	0	9	0 + 9 = ___
2 x 9 =	1	8	1 + 8 = ___
3 x 9 =	2	7	2 + 7 = ___
4 x 9 =	___	6	___ + 6 = ___
5 x 9 =	___	___	___ + ___ = ___
6 x 9 =	___	___	___ + ___ = ___
7 x 9 =	___	___	___ + ___ = ___
8 x 9 =	___	___	___ + ___ = ___
9 x 9 =	___	___	___ + ___ = ___
10 x 9 =	___	___	___ + ___ = ___

Complete the multiplications before doing the subtractions (as in the first example).

1 x 2 − 0 x 3 = 2 − 0 = 2

2 x 3 − 1 x 4 = 6 − ___ = ___

3 x 4 − 2 x 5 = ___ − ___ = ___

4 x 5 − 3 x ___ = ___ − ___ = ___

5 x 6 − 4 ___ = ___ − ___ = ___

6 x 7 − ___ = ___ − ___ = ___

7 x ___ − ___ = ___ − ___ = ___

8 ___ − ___ = ___ − ___ = ___

9 ___ − ___ = ___ − ___ = ___

1	2	3	4	5	6	7
x 1	x 2	x 3	x 4	x 5	x 6	x 7
1	4					
3						
2						

Carry out the indicated multiplications. Record results on black lines.

On the blue lines, record the differences between the two numbers written above to the right and above on the left.

2	3	4	5	6	7	8
x 0	x 1	x 2	x 3	x 4	x 5	x 6
0	3					
3						
2						

Use the green lines similarly. On them, record the difference of the numbers written on the blues lines above right and above left.

Compare the two sets of results.

Did you expect what you see?

---

a. (0 x 3) ÷ 2 = ____

b. (1 x 4) ÷ 2 = ____

c. (2 x 5) ÷ ___ = ____

d. (3 x 6) ÷ ___ = ____

e. (4 x) ÷ ___ = ____

f. (5 x) ÷ ___ = ____

g. (6 x) ÷ ___ = ____

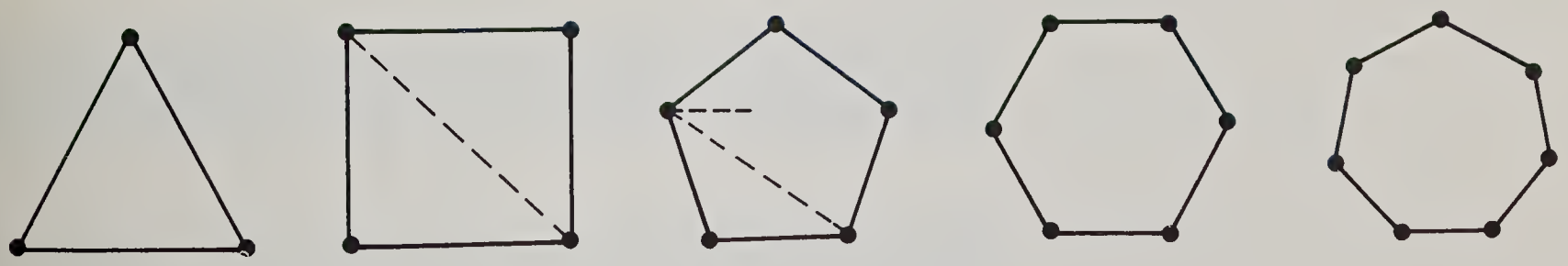
h. (7 x) ÷ ___ = ____

Find the results in the group of examples on the left. Record them under the appropriate blue letters.

a.	b.	c.	d.	e.	f.	g.	h.
0	2						
2							

On the blue lines, record the differences of the numbers written above right and above left. Same for green lines.

Draw all the straight line diagonals you can connecting the dots in each of the figures below. Record number of diagonals on black lines.



Number of diagonals					
Difference of numbers above					
Difference of numbers above					

Have you seen a similar result before?

Anne gave each member of the class a bag and some counters. She said, "Think of a number. Put that number of counters in the bag, and tie it up.

"Put 3 counters by the bag. Now, if you tell me how many counters you have used, I will tell you what's in the bag."

Game I

	Betty	Jim	Ruth	Ted	Ben
Think of a number. Put that many counters in the bag.	5				
Put 3 counters by the bag. How many counters all together?	8	6	12	17	3

Betty thought of 5; Jim thought of 3; Ruth thought of 9; Ted, 14; and Ben, zero.

"That's easy," Ben complained.

"All right," Anne continued, "you will each need extra bags for the next games."

Game II

Think of a number. Put that many counters in the bag.					
Add 5. You now have the bag and extra counters.					
Double it. You now have two bags and 10 extra counters. How many counters all together?	18	32	40	44	10

Game III

Think of a number					
Double it					
Add 6					
Divide by 2. How many counters have you all together?	15	11	3	13	4

Game IV

Think of a number					
Add 7					
Add the number you thought of first.					
Subtract 5. How many have you all together?	16	22	52	2	4

"That game," Ben said, "reminds me of the trick on the first page of surprise. I can make them up myself, now. Here is an example:

"Think of a number.
"Double it. Then add six.
"Divide by two.
"Take away the number you first thought of."

"Your result is 3!"

Trick I

Think of a number	
Double it	
Add 6	
Divide by 2	
Take away number thought of first	

Your Result is 3.

Ben explained it this way:

"Think of a number. Put that many counters in a bag.

"Double it. Now you have two bags with as many counters in each as the number you thought of first.

"Add six. Now you have two bags plus six extra counters.

"Divide by two. You remove one of the bags and three of the counters, leaving a bag and three counters.

"Now, take away the number you thought of first.

Your secret number disappears from the scene, no matter how many counters were in the bag."

"Here are two more tricks," Ben volunteered. "They are longer, but they use the same idea."

The numbers Anne and Bill thought of are shown. Also try a number you think of yourself.

Trick II

		Anne	Bill	You
Think of a number		2	8	
Add 3		5		
Triple the result				
Take away 1				
Take away no. thought of first				
Divide by 2				
Take away no. thought of first				

Trick III

		Anne	Bill	You
Think of a number		10	5	
Double it		20		
Double it again				
Take away no. thought of first				
Add 6				
Divide by 3				
Take away no. thought of first				

Next day, Jerry said he had made up some of Ben's tricks with larger numbers.

"Instead of drawing counters, I'm going to use numbers to keep track of the extra counters."

(Below is the number Ben thought of. Include a number you think of.)

Trick IV

		Ben	You
Think of a number	8	19	
Add 50	8 + 50		
Double it	8 8 +100		
Add 48	8 8 +148		
Divide by 2	8 + 74		
Take away no. thought of first			

"Now," Jerry went on, "I'm going to have to draw large numbers of bags unless we can all agree that all I need to do is draw one bag and tell you how many I mean."

Trick V

		Ben	You
Think of a number	8	7	
Multiply by 10	10 8		
Add 130	10 8 +130		
Divide by 5	2 8 + 26		
Divide by 2	8 + 13		
Take away no. thought of first			

"Those bags are hard to draw," Jerry complained. "So, I'm going to erase the top and bottom of the bag. All there is left is X which looks very much like the letter x. So, I will write $3X + 6$ for the 3 bags and six more."

Trick VI

		Ben	You
Think of a number	x	3	
Add 47	x + 47	50	
Double it	2x + 94		
Take away 75	2x + 19		
Triple it	6x + 57		
Subtract number thought of first	5x + 57		
Add 18	5x + 75		
Divide by 5	x + 15		
Take away no. thought of first			

Finish the picture and use Jerry's shorthand to show the following trick:

Trick VII

Words	Pictures	Shorthand
Think of a number	8	x
Add 4	8 ••••	x + 4
Double it	8	
Subtract 2		
Add number thought of first		
Divide by 3		
Take away no. thought of first		

Fred has an IDEA.

"If you show me the results of any line of the games or tricks, I'll tell you the number each player was thinking of to start with. The only exception would be the last line of Ben's puzzle. There has to be at least one bag left in the picture."

Fred did it. Can you discover his method?

Game II — page 102

Think of a number						
3rd line →	8 8 ●●●●●	18	32	40	24	10

Game III — page 102

Think of a number						
3rd line →	8 8 ●●●	30	22	6	26	8

Game IV — page 102

Think of a number						
4th line →	8 8 ●●	16	22	52	2	4

Trick II — page 103

Think of a number			
5th line	8 8 ●●●●	12	24

Trick III — page 103

Think of a number			
5th line	8 8 8 ●●●	36	21

Trick IV — page 104

Think of a number			
3rd line	8 8 + 100	138	

Trick V — page 104

Think of a number			
4th line	2 8 + 26	40	

Trick VI — page 104

Think of a number			
7th line	5X + 75	90	

Trick VII — page 104

Think of a number			
5th line	3X + 6	18	33

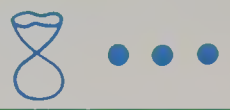
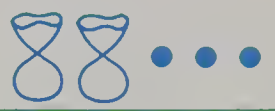
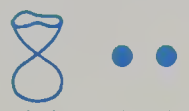
Two Extras

Think of a number			
	4X + 7	11	35

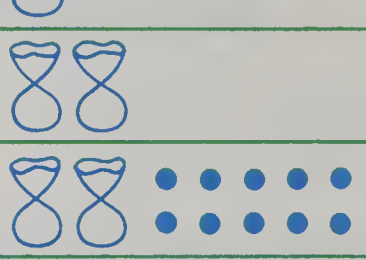
Think of a number			
	3X + 150	225	150

Some problems Fred gave the class. Complete the following records:

Words	Pictures	Examples	
Think of a number			
Multiply by 2			
Multiply by 2 again		20	
Add 6			
Take away number thought of first			33
Divide by 3			
Add 8			
Take away number thought of first			

Words	Pictures	Examples	
			
			15
		13	

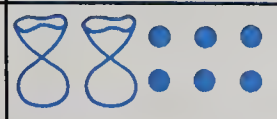
Words	Shorthand	Examples	
Think of a number	X		
Add 9			
Double	2X + 18		26
Add number thought of first			
Divide by 3			
Add 10		26	
Double			
Take away number thought of first			
Subtract 25			
Take away number thought of first			

Pictures	Shorthand	Examples	
			40
		14	

Pictures	Shorthand	Examples	
		4	7
		12	21
		24	33
		20	26
		10	13
		6	6

WANTED: Endings for these Tricks

Trick A

Words	Pictures	Shorthand	Two Examples	
Think of a number		x	9	
Add 4 to it		$x+4$		
Double it		$2x+8$		
Take away 2		$2x+6$		

Can you finish it in two or three steps? In the step before the end, there should not be more than a single bag. When you say, "Take away the number you thought of first," that gets rid of the bag.

Trick B

Words	Pictures	Shorthand	Two Examples	
Think of a number		x		
Double it		$2x$		
Add 9				
Add number thought of first				

Trick C

Words	Shorthand	Example
Think of a number	x	
Add 6	$x+6$	
Double it		

Trick D

Words	Shorthand	Example
Think of a number	x	
Triple it	$3x$	
Add 8		
Take away number thought of first		

Fred had another Idea!

"If you show me part of a record of the kind of trick and games we've been playing, I can often figure out the rest. You don't have to show me the words or pictures or shorthand.

"Here is a sample of the ways two tricks might get started. I'll explain my idea as we go along."

Words	Pictures	Shorthand	Anne	Ben	Mat	Jane	Ruth	Joe	Betty
Think of a number	8	X	5	3	1	11		0	25
			10	8		16	12	5	

"Consider Anne's results. She thought of 5. By applying some rule not shown, she was led to 5. The rule might be 'double it' or, perhaps, 'add 5.' Ben thought of 3. 'Double it' would lead to 6. But his result is 8. However, 'add 5' would explain the results of both Anne and Ben.

"The rule 'add 5' would also explain the results recorded for Jane and Joe. Applying the rule to the numbers Mat and Betty thought of, they would lead to 6 and 30. And we can see that the number Ruth thought of must have been 7, because $7 + 5 = 12$.

"Now we can draw the pictures and show the idea in shorthand."

Here are the first steps of another trick:

Words	Pictures	Shorthand	Anne	Ben	Mat	Jane	Ruth	Joe	Betty
Think of a number	8	X	0	100			10	12	25
			0		21	33	30		75
				315				51	90

"From Anne's results in the first 2 rows, I know the rule must be 'add 0' or it must involve multiplication. Ruth thought of 10 and the rule led her to 30. It must be 'multiply by three' or 'triple the number you thought of.'

"That would also explain Betty's results. Now I can draw pictures and write the shorthand for the second row. I can also find out the missing items in the first and second lines."

Next, consider the second and third rows.

Words	Pictures	Shorthand	Anne	Ben	Mat	Jane	Ruth	Joe	Betty
Think of a number	8	X	0	100	7	11	10	12	25
Triple it	8 8 8	3X	0	300	21	33	30	36	75
				315				51	90

"Ben's results suggest right away that the rule might be 'add 15.' That rule explains the results given for Joe and Betty. Now I can complete the third line."

Here is the record of a trick Fred says he can reconstruct:

Words	Pictures	Shorthand	Anne	Ben	Mat	Jane	Ruth	Joe	Betty
a. Think of a number			11	7	3		1		9
b.			22	14	6			12	
c.							8	18	24
d.			14		6	11	4		
e.				3		3		3	3

"Consider lines a and b," Fred advises.

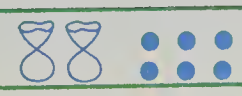
a. Think of a number			11	7	3		1		9
b.			22	14	6			12	

"In line a, I begin by drawing a bag. When anyone thinks of a number, I suppose they put that many counters in the bag. Then I use Jerry's shorthand. I erase the top and bottom of the bag. The part that is left looks like the letter 'X.'

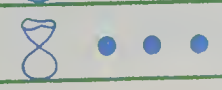
"Next, I consider the results shown for Anne, Ben, and Mat. The rule must be 'double' the number you thought of. So, I write this idea in words, pictures, and shorthand. Now, I can fill in the missing results for Ruth, Joe, and Betty."

Results of line b		2X	22	14	6		2	12	18
c.							8	18	24

By studying the results for Ruth, Joe and Betty, I know the rule must be 'add 6.'

Results of line c		2X + 6	28	20	12		8	18	24
d.			14		6	11	4		

By studying the results shown for Anne, Mat and Ruth, I know the rule is 'divide by 2.'

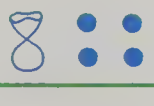
a. Number thought of first		X	11	7	3		1	6	9
d. Results of line e		X + 3	14	10	6	11	4	9	12
e.				3		3		3	3

"Consider the lines a, d, and e. The results for Ben, Joe and Betty give the clue. The rule must be: from the results in line d, 'take away the number you thought of first.' Now, you can complete reconstruction of the entire record."

Ben made up some challenging examples. Can you reconstruct them?

Words	Pictures	Shorthand	Anne	Ben	Mat	Jane	Ruth	Joe	Betty
Think of a number		X	10			5		8	
					12	8		11	
			26	6	24		30		
				10			34		
				5		10			20
				5					5

Words	Pictures	Shorthand	Anne	Ben	Mat	Jane	Ruth	Joe	Betty
Think of a number									
			12	0	30		75		
		3 X + 6	18	6	36				
				6	26	16	56		
								103	22
Subtract number thought of first		3							

Pictures	A	B	C	D
				8
		17		18
	7	11	15	
	4		4	

Pictures	I	J	K	L
	2	4		0
		9	20	5
		1		1

↓

Shorthand	E	F	G	H
X				7
		0		
2 X + 20		20	26	34
	26	20		27
			5	
2				

Shorthand	M	N	O	P
X .				
	9		13	
3 X + 12	27			
		38		34
X + 6		19	15	17
6				

Fred repeats his claim:

"Some of these guessing games are becoming more complicated. But, if you give me the shorthand and results in any row, I can still tell the number each player thought of first."

	Shorthand	A	B	C	D	E	F	G	H	I
One row of results	$X+8$	18	19	20	8	100	16	39	29	9
Number each player thought of first	X									

One row of results	$2X+5$	15	35	55	5	7	9	115	91	67
Number thought of first	X									

One row of results	$4X-3$	1	9	17	97	37	397	25	41	57
Number thought of first	X									

One row of results	$10+X$	12	20	11	60	10	21	39	35	17
Number thought of first	X									

One row of results	$3X-6$	15	0	30	24	9	144	36	69	84
Number thought of first	X									

One row of results	$200-X$	195	199	125	200	193	190	50	100	19
Number thought of first	X									

One row of results	$25X+25$	225	125	50	275	150	25	400	200	250
Number thought of first	X									

One row of results	$9+3X+5X$	9	33	17	65	25	49	97	89	169
Number thought of first	X									

One row of results	$4X+7-2X$	11	7	29	9	21	27	107	57	17
Number thought of first	X									

One row of results	$9+2X+5X$	65	44	16	58	9	72	86	184	79
Number thought of first	X									

Anne explains an interesting and useful result.

"If I think of any number, I can always get the same results in two ways that seem very different.

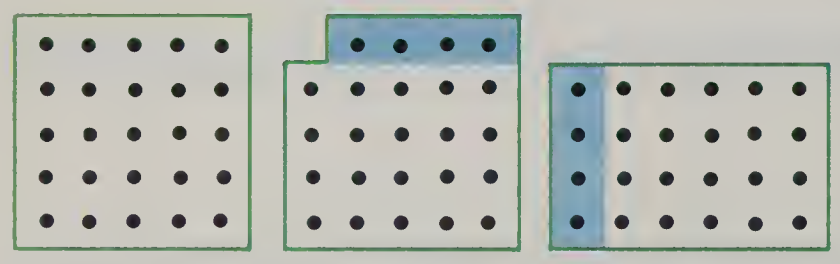
"I'll keep track of these ways in shorthand — but I'm not going to use bags or 'χ' because 'n' or any other mark would be just as good, as long as we agree it means the number you thought of first."

(Try some examples of your own)

Words	Shorthand	Ben	Al	Mary	Jane	A	B	C	D
Think of a number	n	3	5	11	4				
Multiply your number by itself	n · n	9							
1 less than that	n · n − 1	8							
1 more than your number	n + 1	4							
1 less than your number	n − 1	2							
Multiply these together	(n+1) · (n−1)	8							

"For any number you can think of, it is true that $n \cdot n - 1 = (n + 1) \cdot (n - 1)$

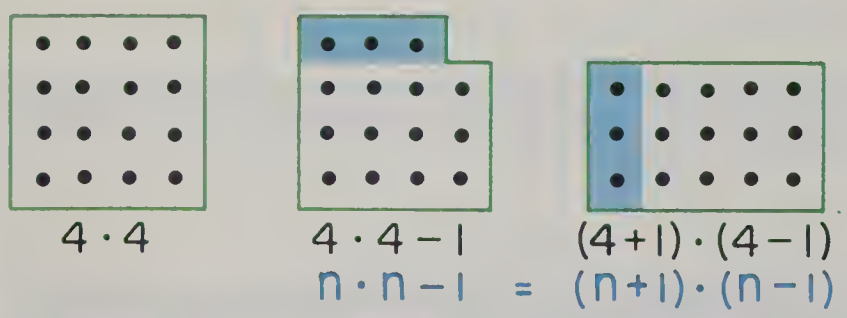
"You can make a picture to show why this happens. For example, the number Al thought of was 5."



5 · 5 | 5 · 5 − 1 | (5 + 1) · (5 − 1)

"After you take one dot away from 5 rows of 5 dots, you have just enough to make 4 rows of 6 dots.

"Jane's example works the same way."



4 · 4 | 4 · 4 − 1 | (4 + 1) · (4 − 1)

n · n − 1 = (n+1) · (n−1)

11 · 11 | 11 · 11 − 1 = (11+1) · (11−1)

"This is useful as a short cut. When I want to know the answer to 21 × 19, I think of this as (20 + 1) · (20 — 1); and I know that I can find the result by finding 20 · 20 — 1. It is 399. You can use this same method in the following:

31 x 29 = ____ 51 x 49 = _____

13 x 11 = ____ 101 x 99 = _____

41 x 39 = ____ 71 x 69 = _____

"It is also a short cut when I want to find one less than 21 × 21. I know that 21 · 21 — 1 = 21 + 1 × 21 — 1 or 22 × 20. The result is 440."

31 · 31 − 1 = ____ 51 · 51 − 1 = _____

41 · 41 − 1 = ____ 101 · 101 − 1 = _____

"So, of course, I can use this short cut in the following:"

31 · 31 = ____ 51 · 51 = _____

21 · 21 = ____ 41 · 41 = ____

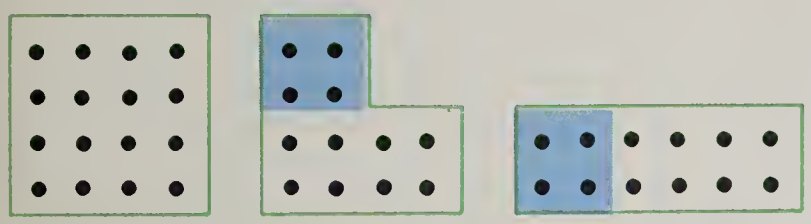
"This result is quite limited as a short cut," Anne continued. "But it suggests further experiments that lead to similar short cuts. Let's agree again that 'n' is shorthand for any number you think of. However, 'a' is just as useful."

(Examples of your own)

Words	Shorthand	Ben	Al	Mary	Jane	E	F	G	H
Think of any number	n	4	3	2	5				
Multiply it by itself	n · n	16							
4 less than that	n · n − 4								
2 more than your number	n + 2	6							
2 less than your number	n − 2	2							
Multiply these together	(n+2)·(n−2)								

For any number you can think of, it's true that $n \cdot n - 4 = (n + 2) \cdot (n - 2)$.

"You can draw a picture to show why this happens. For example: Ben thought of 4.



4 · 4 | 4 · 4 − 4 | (4−2)·(4+2)

"Jane thought of 5. You can draw a picture to show the results."

5 · 5 | 5 · 5 − 4 | (5−2)·(5+2)

"You can use this result as a short cut in the examples to the right."

22 · 22 − 4 = _____

32 · 32 − 4 = _____

18 · 22 = _____

48 · 52 = _____

(Examples of your own)

Words	Shorthand	Ben	Al	Mary	Jane	I	J	K	L
Think of any number	n	7	4	3	5				
Multiply it by itself	n · n								
9 less than that	n · n − 9								
3 more than your number	(n+3)								
3 less than your number	(n−3)								
Multiply these together	(n+3)·(n−3)								

For any number you can think of, it is true that $n \cdot n - 9 = (n + 3) \cdot (n - 3)$.

Anne gave another example:

(Examples of your own)

		Ben	Al	Mary	Jane	M	N	O	P
Think of any number	n	5	10	4	8				
Multiply it by itself	$n \cdot n$								
16 less than that	$n \cdot n - 16$								
4 more than your number	$n + 4$								
4 less than your number	$n - 4$								
Multiply these together	$(n+4)\cdot(n-4)$								

"For any number you think of, it's true that $n \cdot n - 16 = (n + 4) \cdot (n - 4)$."

"Now, let's summarize our results," Ann suggested. "On this and the last two pages, we have found the following which we can test altogether."

	The number Ben thought of was 5	The number Mary thought of was 8
A. $n \cdot n - 1 = (n+1)\cdot(n-1)$	$25 - 1 = 6 \cdot 4$	$64 - 1 = 9 \cdot 7$
B. $n \cdot n - 4 = (n+2)\cdot(n-2)$	$25 - 4 =$	$64 - 4$
C. $n \cdot n - 9 = (n+3)\cdot(n-3)$	$25 - 9 =$	
D. $n \cdot n - 16 = (n+4)\cdot(n-4)$		

"Then the following group of statements ought to be true. Let's test them."

	If the number thought of is 9	If the number thought of is 4
E. $n \cdot n = (n+1)\cdot(n-1) + 1$	$81 = 10 \cdot 8 + 1$	$16 = 5 \cdot 3 +$
F. $n \cdot n = (n+2)\cdot(n-2)+4$	$81 = 11 \cdot 7 +$	
G. $n \cdot n = (n+3)\cdot(n-3)+9$		
H. $n \cdot n = (n+4)\cdot(n-4)+16$		

"The following statements must also be true," Anne asserted.

	If the number thought of is 7	If the number thought of is 10
I. $(n+1)\cdot(n-1) = n \cdot n - 1$	$8 \cdot 6 = 49 -$	
J. $(n+2)\cdot(n-2) = n \cdot n - 4$		
K. $(n+3)\cdot(n-3) = n \cdot n - 9$		
L. $(n+4)\cdot(n-4) = n \cdot n - 16$		

Finally, Anne ties these ideas together into a handy shortcut.

"There is a pattern running through the examples on the last three pages," Anne explained. "It is something like a game that might be called 'Think-of-Any-Two-Numbers.' Here's the way it works."

Think-of-Any Two Numbers	Shorthand	Ben	Ruth	Ellen	Roy	(Your own examples) A	B	C
Your bigger number	b	7	6	10	15			
Your smaller number	s	3	4	7	10			
The sum of your numbers	b + s	10						
The difference of your numbers	b − s	4						
These multiplied together	(b+s)·(b−s)	40						
The bigger number thought of times itself	b · b	49						
The smaller number thought of times itself	s · s	9						
Subtract these	b · b − s · s	40						

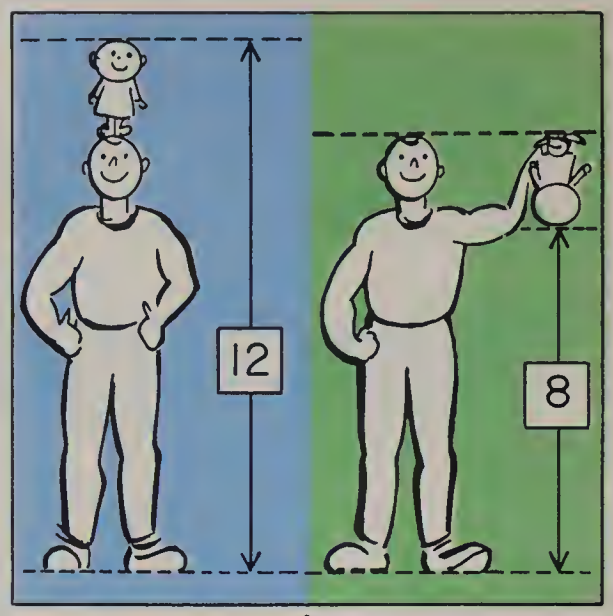
"For any pair of numbers you think of, (b + s) · (b — s) = b · b — s · s."

"Many pairs of factors," Anne explained, can be rewritten in the form of (b + s) (b — s) and in the form of b · b — s · s. Often, these forms point to important shortcuts. Sometimes they do not help at all."

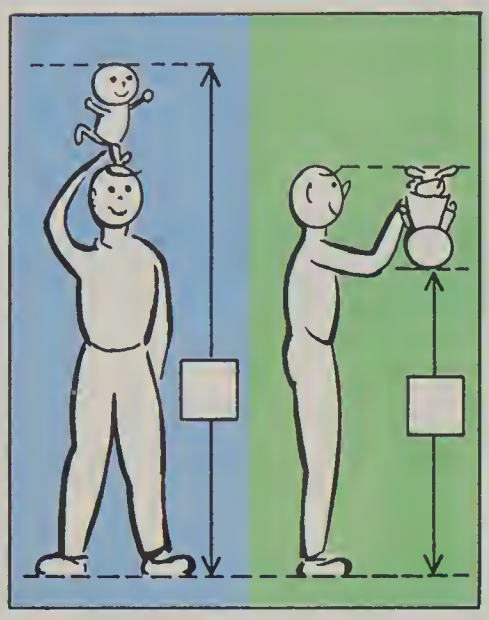
Pair of Factors	Rewritten in form of (b + s) · (b — s)	Rewritten in form of b · b — s · s			
12 · 8 =	(10 + 2) · (10 − 2) =	10 · 10 − 2 · 2 =	100 − 4 =	____	
10 · 16 =	(13 − 3) · ( ) =	=	=		
21 · 19 =	(20 + 1) · ( ) =	=	=		
22 · 18 =	=	=	=		
35 · 25 =	=	=	=		
53 · 47 =	=	=	=		
56 · 44 =	=	=	=		
109 · 91 =	=	=	=		
79 · 81 =	=	=	=		

Make up examples of your own.

Guessing Games in the Land of Giant Acrobats and Dwarf Acrobats.



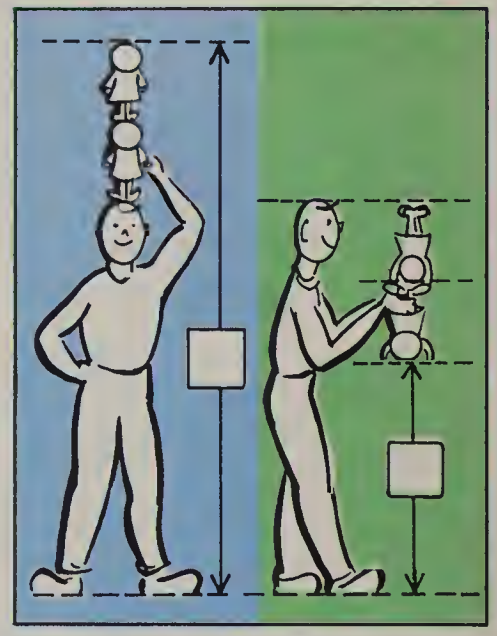
In each team of acrobats, all giants are the same height and all dwarfs are the same height. (All heights given in numbers of feet.) An example: a giant and a dwarf together are 12 feet tall. A giant's length less a dwarf is 8 feet. How tall is each? Ben and Anne imagine different results:

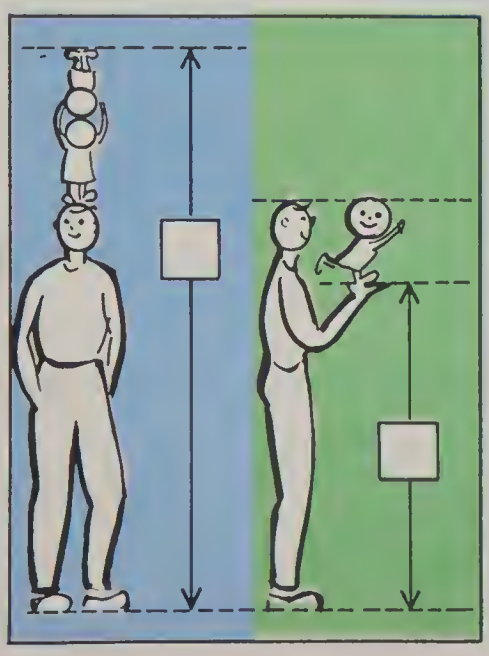


Height in number of feet of:	Shorthand	Example	Ben	Anne
a giant and a dwarf	g + d	12		15
a giant less a dwarf	g – d	8		13
a giant	g		28	
a dwarf	d		5	

Fred, Dot, and Mary made up imaginary heights for members of another imaginary team of acrobats. Guess what heights they imagined.

Height in number of feet of:	Shorthand	Fred	Dot	Mary
a giant and two dwarfs	g + 2d	18		45
a giant less two dwarfs	g –		16	25
a giant	g		20	
a dwarf	d	3		



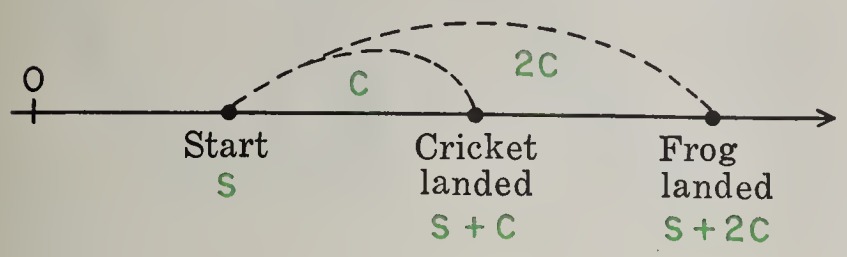


Al, Jo, and Nan imagined other heights for the arrangement of acrobats shown on the left.

Height in number of feet of:	Shorthand.	Al	Jo	Nan
a giant and two dwarfs	g +	29	50	20
a giant less a dwarf	g –	23		14
a giant	g		40	
a dwarf	d			

Ruth's Guessing Game . . . . . (but she wasn't really guessing)

Ruth drew a numberline on the board. "A cricket and a frog started at the same point. The frog jumped twice as far as the cricket in the direction away from 0.

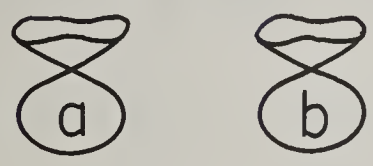


Each of you must imagine where they started and how far the cricket jumped. Tell me where they landed — and I'll tell you where you imagined they started and the direction you imagine each jumped."

	Shorthand	Ben	Anne	Joe	Al	Ellen	Mary	Alice	Betty	Pete
Think of a number that indicates the starting point	S	5								
Think of another number that indicates the size of cricket's jump	C	3								
Frog jumped twice as far	2C	6								
Tell where cricket landed	S + C	8	16	14	7	24	12	14	17	71
Tell where frog landed	S + 2C	11	22	18	14	34	23	21	26	92

Ruth told each player the numbers he had thought of. Can you?

Ruth drew a pair of bags and labeled one "a" for Andy, and the other "b" for Bill. Then she explained that Andy had more marbles in his bag than Bill had.



"Suppose you know how many each had. Use any two numbers you think of. Just tell me how many they have together and how many more Andy has than Bill."

	Shorthand	Ben	Anne	Joe	Al	Ellen	Mary	Alice	Betty	Pete
Think of a number to indicate how many marbles Andy had	a	10								
Think of a number to indicate how many marbles Bill had	b	3								
How many together?	a + b	13	15	1	10	17	28	17	64	132
How many more does Andy have than Bill?	a − b	7	3	1	6	1	12	17	6	34

Again, Ruth told each player the numbers he had thought of. Can you?

(Make up your own story for the next example.)

	Shorthand	Ben	Anne	Joe	Al	Ellen	Mary	Alice	Betty	Pete
Think of a number	m	3								
Think of a second number	n	5								
Sum of the two numbers	m + n	8	4	6	8	9	2	25	18	30
Add second to three times first	3m + n	14	8	16	10	27	4	45	40	60

A Gate Problem

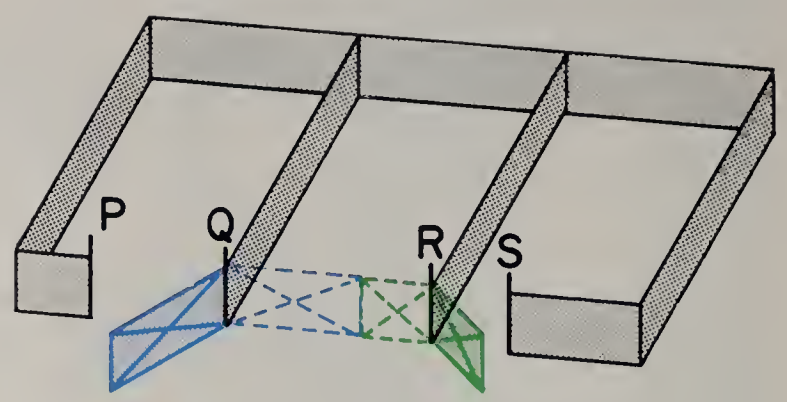
Ruth drew a diagram and explained:

"Each of you pretend you are a farmer with three fields. They are fenced in as shown. You have only two gates — one blue and one green. They may be different lengths. You arrange them as shown.

"The blue gate alone can close in the field on the left; the green one alone can close in the field on the right; the blue and green are both needed to close in the center field. In this case, the gates meet but do not overlap.

"So we can talk about them, I have labeled four posts that are in a line — P, Q, R, and S. Since you know the length of the gates, you also know how to space the posts.

"In the chart below, fill in any two of



the questions marked A, B, C, D, E, F. Unless you trick me by answering a certain pair of questions, I can figure out the length of the gates and the distances between posts.

"And, of course, if you answer the first two questions, my job is easy. Can you find a pair of questions that do not give me enough information?"

What is the number of feet in:	Shorthand	Some of the players: Anne	Ben	Carl	Dan	Ellen	Fred	Gil	Harry	Ivan	June
A. the length of the blue gate	b										
B. the length of the green gate	g										
C. the distance from Q to R		11	19		27			19		26	
D. the distance from P to R		18		41			32		28		15
E. the distance from P to S				52	54	52	46	38			
F. the distance from Q to S			28			39			38	41	

Did you discover the two players that stumped Ruth? They were the ones that answered questions C and E.

Compare all the results in row C with those in row E. Do you notice a pattern? Does it help explain why Don and Gil stumped Ruth?

If some player answered only question C, would you know how he would have to answer question E?

If you were told that the blue gate is twice as long as the green one, can you complete a chart with only 1 question answered?

	Shorthand	K	L	M	N
A___	b				
B___	g		5		
C___	b + g	27			
D___	2b + g				
E___	2b + 2g				66
F___	b + 2g			24	

"Here's another game," Ruth began.

"Each of you think of any two numbers. Then tell me what you get for any two of the six expressions given below. I will tell you what you would get for the other expressions. I will also tell you the two numbers you thought of."

	Shorthand							
	one number	other number						
Player	x	y	x+y	2x+y	x+2y	3x+y	3x+2y	3x+4y
Anne			8	11				
Betty			9			23		
Carl					15		25	
Dan			19		28			
Ellen				31		46		
Fred				19			34	
Gil						10		22

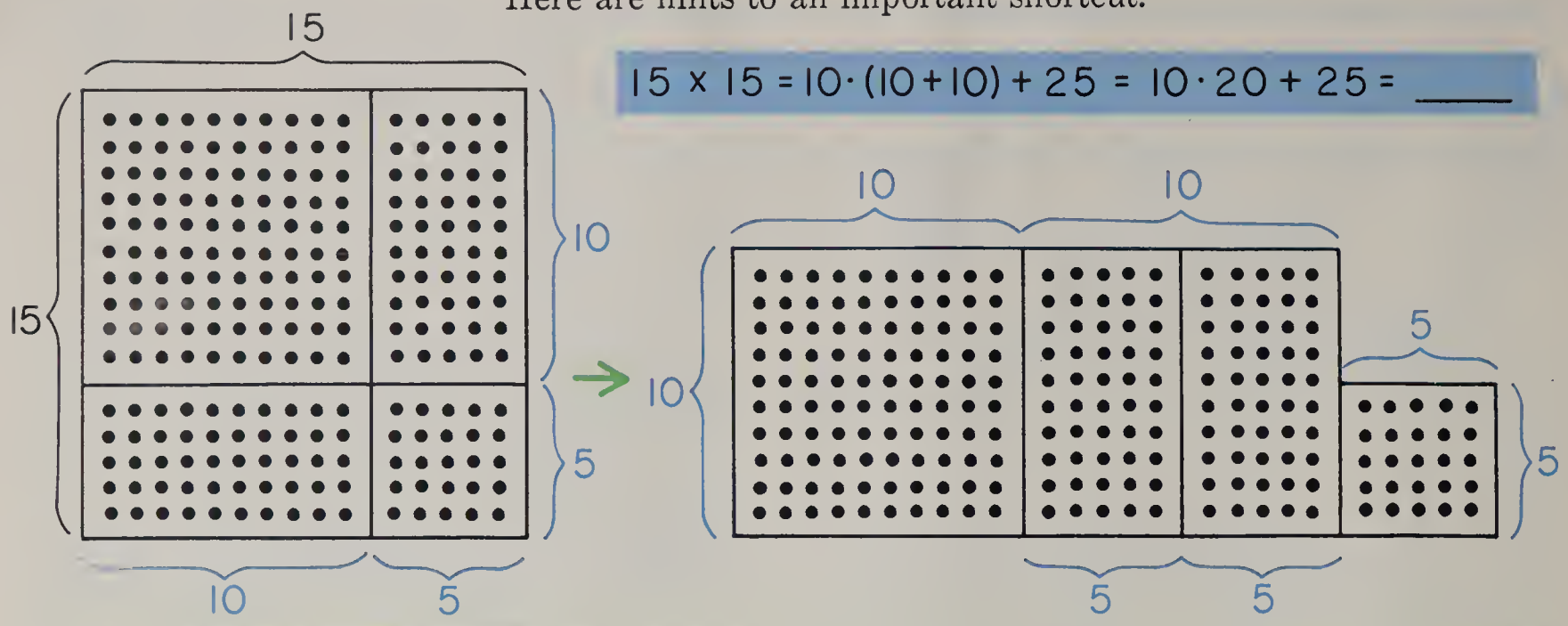
"Think of two other numbers — one larger than the other. The rest of the rules are the same. I will change the expressions. Tell me what you get for any two."

	Shorthand							
	bigger number	smaller number						
	b	s	b+s	b−s	b+2s	2b−s	2b+s	3b−s
Anne			15	5				
Betty				1		10		
Carl				3	6			
Dan						22	26	
Ellen						25		40
Fred			6					18
Gil			11				18	

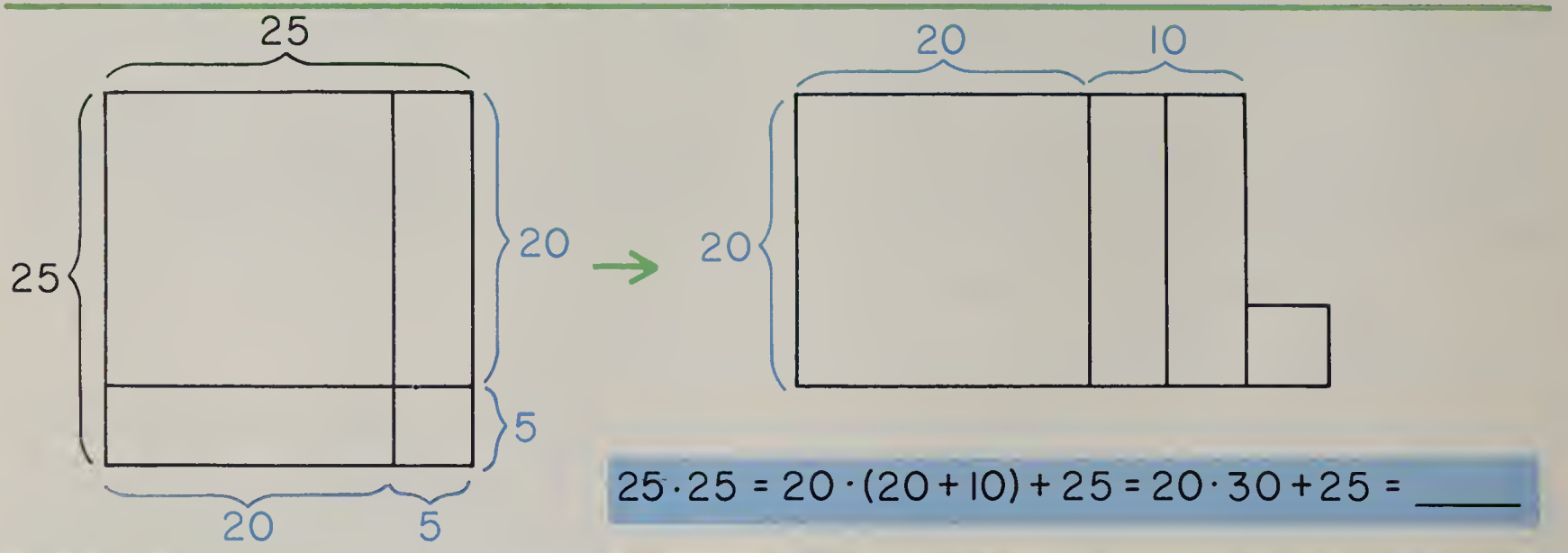
"Again think of two numbers. What do you get as results for two of the five statements? I will tell you the numbers you thought of."

Words	Shorthand	Anne	Betty	Carl	Dan	Ellen	Fred	Gil
Think of a number	p							
Think of a smaller number	q							
The sum of your numbers		8	12		19	29		
The smaller number plus twice the larger number		14				49		
The difference of your numbers			2	7	5		6	
From five times the larger number subtract the smaller number				47			50	35
Three times the larger number plus twice the smaller number								21

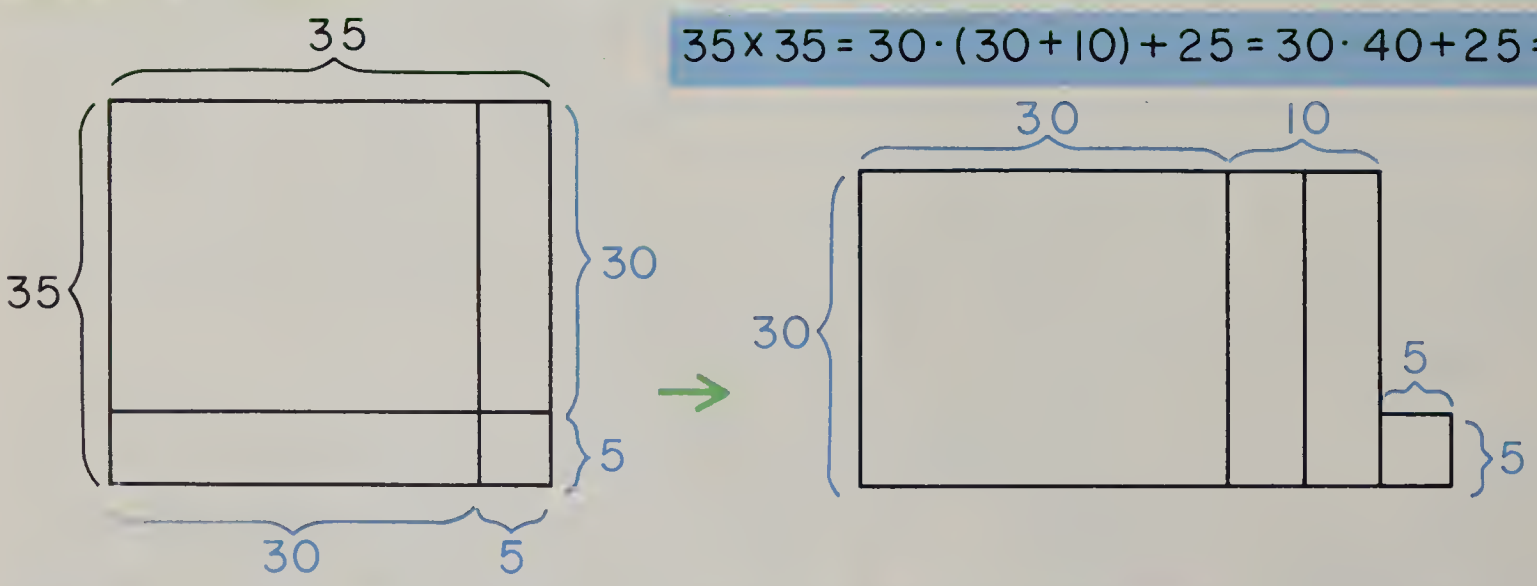
Here are hints to an important shortcut.



$15 \times 15 = 10 \cdot (10+10) + 25 = 10 \cdot 20 + 25 =$ ____



$25 \cdot 25 = 20 \cdot (20+10) + 25 = 20 \cdot 30 + 25 =$ ____



$35 \times 35 = 30 \cdot (30+10) + 25 = 30 \cdot 40 + 25 =$ ____

$15 \cdot 15 = 10 \cdot (10 + 10) + 25 = 10 \cdot 20 + 25 =$ ____

$25 \cdot 25 = 20 \cdot (20 + 10) + 25 = 20 \cdot 30 + 25 =$ ____

$35 \cdot 35 = 30 \cdot (30 + 10) + 25 = 30 \cdot 40 + 25 =$ ____

$45 \cdot 45 = 40 \cdot (40 + 10) + 25 = 40 \cdot 50 + 25 =$ ____

$55 \cdot 55 = 50 \cdot (50 + 10) + 25 =$ _____ = ____

Can you guess the results for:

$65 \times 65 =$ ____

$85 \times 85 =$ ____

Test your guesses.

You can put the answers down as fast as you can write:

15 x 15	25 x 25	35 x 35	45 x 45	55 x 55	65 x 65	75 x 75	85 x 85	95 x 95	105 x 105

Now put this valuable shortcut to work. Combine it with $(b + s) \cdot (b - s) = b \cdot b - s \cdot s$.

$23 \cdot 17 = (20+3) \cdot (20-3)$	$= 20 \cdot 20 - 3 \cdot 3$	$= 400 - 9$	$=$ _____	
$27 \cdot 23 = (25+2) \cdot (25-2)$	$= 25 \cdot 25 - 2 \cdot$	$= 625 -$	$=$	
$31 \cdot 39 = (35+4) \cdot (35-\ \ )$	$= \ \cdot \ - \ \cdot$	$=\ -$	$=$	
$41 \cdot 39 =$	$=$	$=$	$=$	
$24 \cdot 26 =$	$=$	$=$	$=$	
$43 \cdot 47 =$	$=$	$=$	$=$	
$98 \cdot 92 =$	$=$	$=$	$=$	
$101 \cdot 109 =$	$=$	$=$	$=$	
$78 \cdot 82 =$	$=$	$=$	$=$	
$49 \cdot 61 = (55+\ \ ) \cdot (55-\ \ )$	$=$	$=$	$=$	

Combine this new short cut with Ben's Do-It-To-Both rule.

$25 \cdot 26 = 25 \cdot (25+1)$	$= 25 \cdot 25 + 25 \cdot 1$	$=\ +$	$= 650$
$35 \cdot 38 = 35 \cdot (35+\ \ )$	$=\ \cdot\ +\ \cdot$	$=\ +$	$=$
$25 \cdot 24 = 25 \cdot (25-1)$	$= 25 \cdot 25 - 25 \cdot 1$	$=\ - 25$	$=$
$45 \cdot 43 = 45 \cdot (\ \ -\ \ )$	$=\ \cdot\ -\ \cdot$	$=\ -$	$=$
$39 \cdot 35 = 35 \cdot (\ \ +\ \ )$	$=\ \cdot\ +\ \cdot$	$=\ +$	$=$
$75 \cdot 77 = \ \cdot (\ \ \ \ )$	$=\ \cdot\ +\ \cdot$	$=\ +$	$=$

Now take most of the steps without writing them down — if you can.

34 x 35	33 x 37	53 x 47	47 x 45	65 x 66	87 x 93	71 x 59
1225						
− 35						
1190						

$35 \cdot 35 =$	$35 \cdot 36 =$	$34 \cdot 36 =$	$31 \cdot 39 =$
$45 \cdot 45 =$	$46 \cdot 45 =$	$48 \cdot 42 =$	$46 \cdot 44 =$

Another guessing game: WHAT'S MY RULE?

"I have a new guessing game," Bill announced one day.

"I have a secret rule. Someone calls out a number. I will use my rule and answer by saying a number. Can you guess my rule?" To himself, Bill said: whatever number anyone calls out, I will answer with a number that is 1 more.

"7," Anne called out.

"8," Bill answered.

"3," Jane called.

"4," Bill answered.

"15," Al said, "and I think you are going to answer by saying 16."

"That's right, you know my rule already. Someone call out another number."

"20," someone called out.

"How many know what my answer will be?", Bill asked.

Almost everyone did. It would be 21, because 21 is 1 more than 20.

"I have a new rule," Bill announced. To himself, he thought: whatever number anyone says, I will double it.

"2," Helen began.

"4," Bill answered.

"9," Jim called out, "and I think you will answer by saying 11." He thought Bill was adding 2 to each number called out.

"No, I'm going to say 18."

"6," Anne called out, "and I think you are going to answer by saying 12."

"Right."

By this time, everyone was sure of the secret rule.

"I have a new rule," Bill stated.

"16," Ben started off.

"8," Bill answered.

"12," Anne said and, because she thought Bill was dividing each number called out by 2, she added, "I think you plan to answer by saying 6."

"No. My answer is 4."

"18."

"10," Bill answered.

"3," Fred called out, smiling.

"Say something larger," Bill asked.

"7," Fred called out and smiled again.

"Still too small," Bill complained.

"All right, 8, then — and I'm quite sure your answer will be 0."

"Right. You have it."

"28," someone called out.

"20," Bill answered. Now everyone had discovered the secret.

Below are records of other examples of Bill's guessing game.

No. someone calls out	3	8	1	0	11	15	19
Bill's answer	9	24	3	0			

No. someone calls out	7	3	11	63	0	87	13
Bill's answer	17	13	21				

No. someone calls out	1	5	7	0	15	2	9
Bill's answer	9	5	3	10	Too Big		

No. someone calls out	2	10	14	7	16	0	8
Bill's answer	1	5	7	$3\frac{1}{2}$			

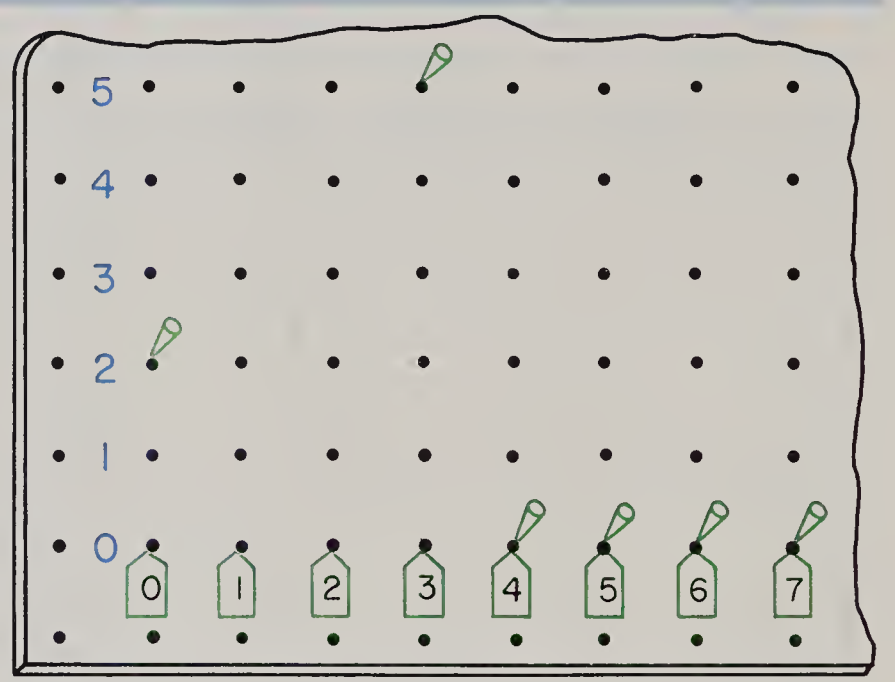
No. someone calls out	2	4	1	3	0	5	7
Bill's answer	4	16	1	9			

Bill finds a way to keep score when playing WHAT'S MY RULE?

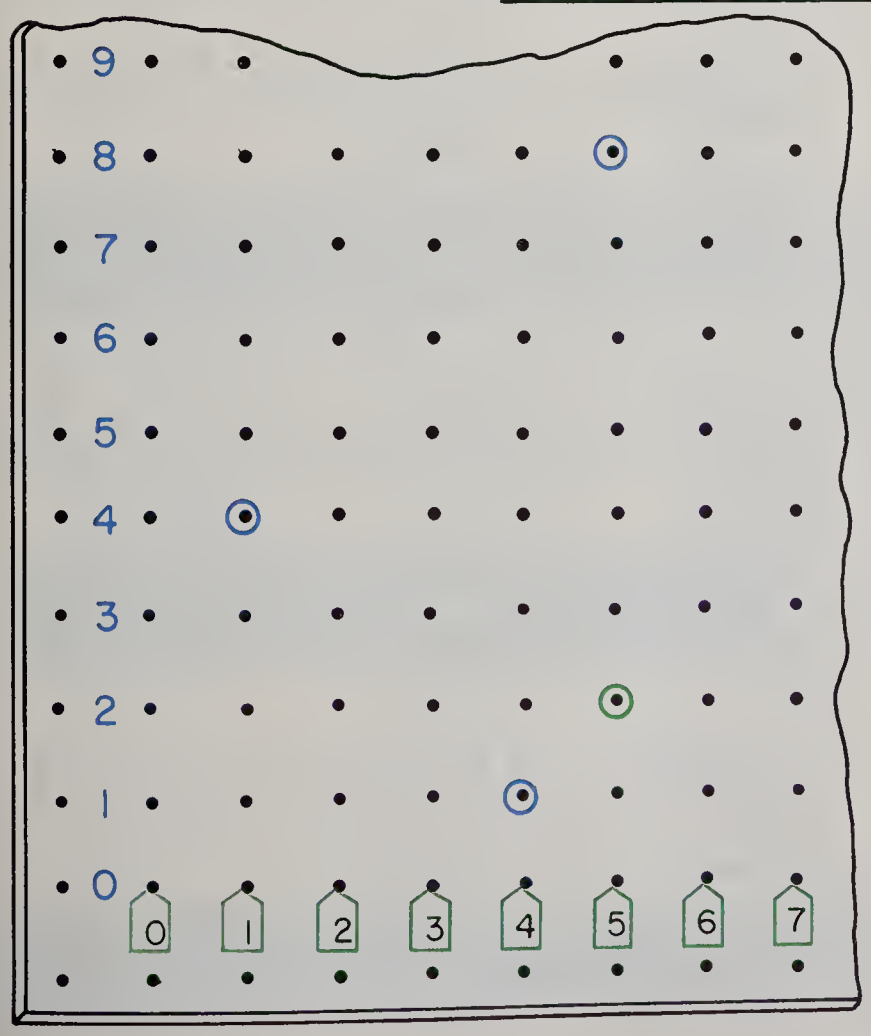
Using peg board, green pegs and tags, Bill worked out a way of keeping track of the numbers people call out and the number he answers after applying his rule.

If someone calls out "2" and Bill answers "3," he finds the peg near the tag with 2 written on it. He moves the peg up to the row of holes opposite 3 on the left side of the board. It would look like the picture to the right.

"I'm thinking of a rule," Bill began.
"3," Ben called out.
"5," Bill answered. He recorded the answer. ⟶
"0," Helen called out.
"2," Bill answered, and recorded the answer. ⟶
"1," Anne called out, "and I think you are going to answer 3." ⟶
"That's right. You know my rule."
"If I say 2, will you say 4?", Al asked.
"Right. You know the rule also." ⟶



The first two results are recorded above. Can you record the others?



On the left, keep a record of the games shown below. You need to discover the secret rule in each case to finish it.

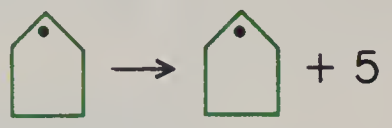
Instead of drawing pegs, circle the place where the peg would go. The first two answers are already indicated for one game, and the first answer for the second is indicated.

No. someone calls out	1	5	4	0	6	3	2
Bill's answer	4	8	7				

No. someone calls out	5	3	6	0	4	1	2
Bill's answer	2	4	1				

Bill explains a shorthand way to write down rules after they are discovered.

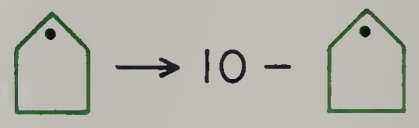
"When someone calls out a number, I imagine it as written on a tag at the bottom of the scoreboard. If my rule is to answer by saying a number that is five more than the number called out, I think this way:



"So, the green tag holds a place for any number you think of. I read that shorthand as:

'Tag goes to tag plus 5.'

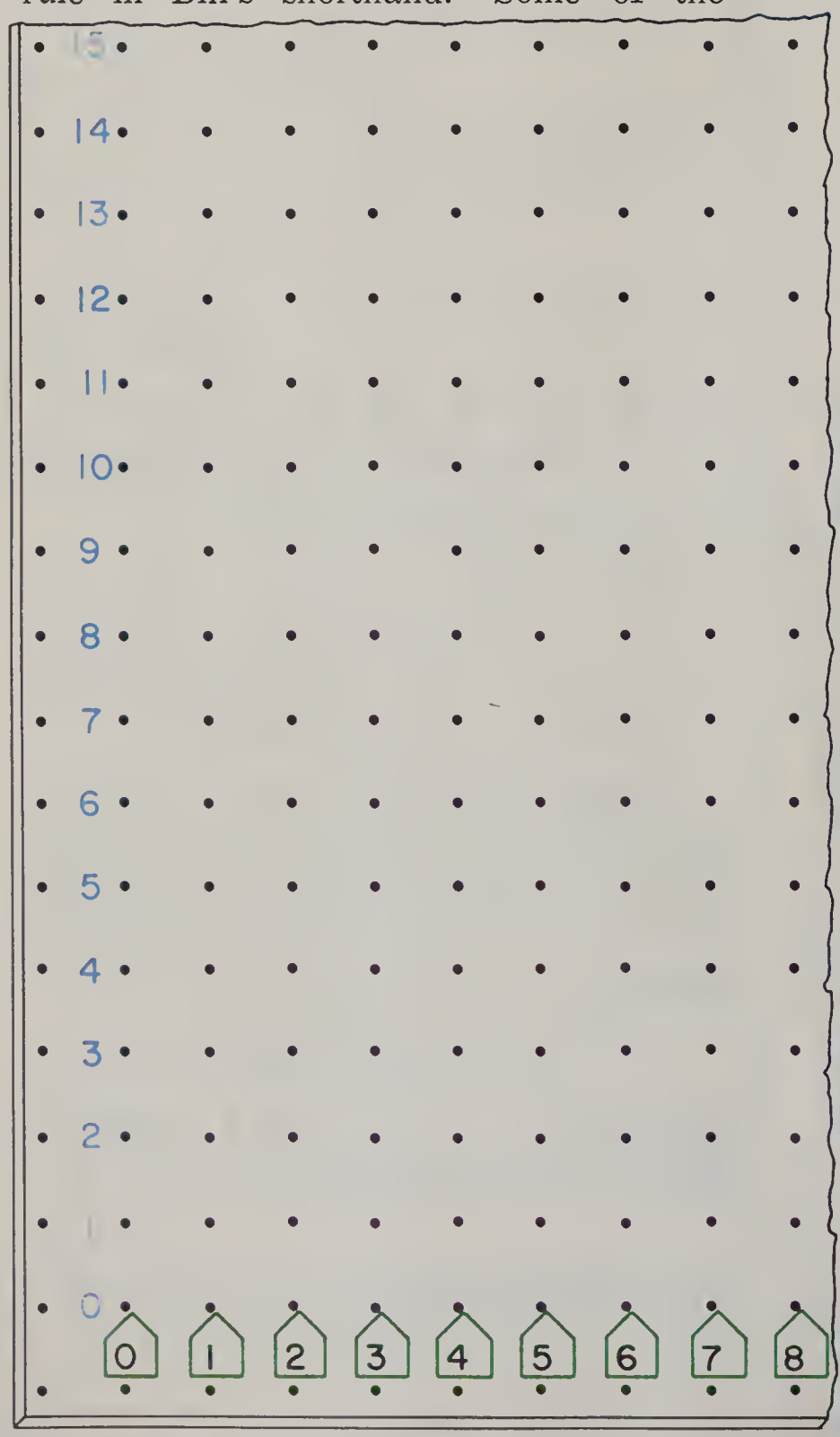
"Here's another rule. How much less than 10 is each number called out? In shorthand and words:



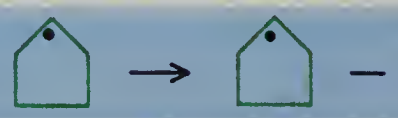
'Tag goes to 10 minus tag.'

Here are some games to finish. Record the results on the scoreboard and write the rule in Bill's shorthand. Some of the numbers called out and some of Bill's answers may be so large you can't record them on the small section of the scoreboard below. In that case, you will have to imagine where the pegs would go.

(Use a different color, if you can, in showing where the pegs would go for each different game.)



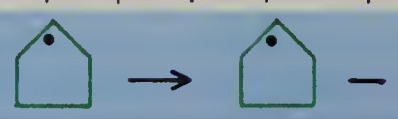
No. someone calls out	5	8	2	1	4	7	20
Bill's answer	3	6	0	too small			

Bill's Rule: 

Numbers called out	3	8	0	6	5	19	7
Bill's answer	9	14					

Bill's Rule:

Numbers called out	7	5	8	4	1	6	83
Bill's answer	3	1			too small		

Bill's Rule: 

Numbers called out	2	4	8	0	3	5	1
Bill's answer							

Bill's Rule:

Bill changes the labels on his scoreboard.

"This far, we have been limited to whole numbers. I have changed the tags on the scoreboard so we can record some fractions . . . those fractions with denominator 2 and those that can be written with denominator 2. For example: $3\frac{1}{2}$ can be written $3 + \frac{1}{2}$ or $\frac{7}{2}$.

"Watch for patterns in the results as we record them on the scoreboard. Sometimes the patterns will be very much like those on the previous page. Some will be different.

"Here are some game starts that you can couple when you discover the secret rule."

(Remember: ⌂ is shorthand for the number someone calls out.)

⌂	Bill's answers
1	5
4	2
3	3
6	
$1\frac{1}{2}$	

Bill's Rule ⌂ → − ⌂

⌂	Bill's answers
4	0
$2\frac{1}{2}$	$1\frac{1}{2}$
1	3
2	
$3\frac{1}{2}$	

Bill's Rule ⌂ → − ⌂

⌂	Bill's answers
$\frac{1}{2}$	1
3	$3\frac{1}{2}$
0	$\frac{1}{2}$
4	
$15\frac{1}{2}$	

Bill's Rule ⌂ → + ⌂

⌂	Bill's answers
$\frac{1}{2}$	$1\frac{1}{2}$
$1\frac{1}{2}$	$\frac{1}{2}$
2	0
0	
1	

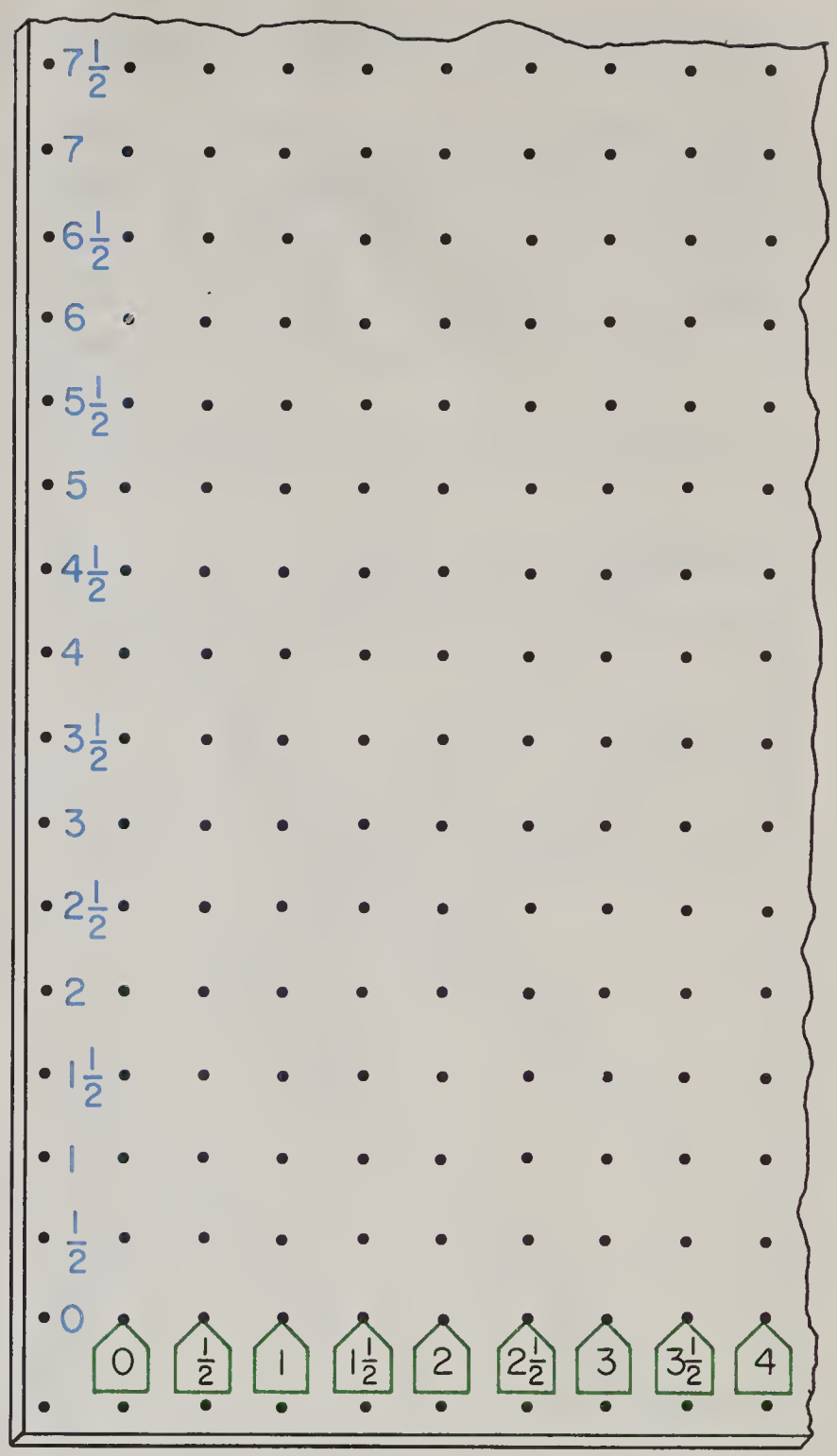
Bill's Rule ⌂ → − ⌂

⌂	Bill's answers
2	1
4	3
$1\frac{1}{2}$	$\frac{1}{2}$
3	
$2\frac{1}{2}$	

Bill's Rule ⌂ → ⌂ −

⌂	Bill's answers
0	$2\frac{1}{2}$
$\frac{1}{2}$	2
1	$1\frac{1}{2}$
$1\frac{1}{2}$	
2	

Bill's Rule ⌂ → − ⌂



⌂	4	$\frac{1}{2}$	3	0	$1\frac{1}{2}$	2	$7\frac{1}{2}$
Bill's answers	4	$7\frac{1}{2}$	5				

Bill's Rule ⌂ →

⌂	4	$1\frac{1}{2}$	3	2	$3\frac{1}{2}$	7	19
Bill's answers	$2\frac{1}{2}$	0	$1\frac{1}{2}$				

Bill's Rule ⌂ →

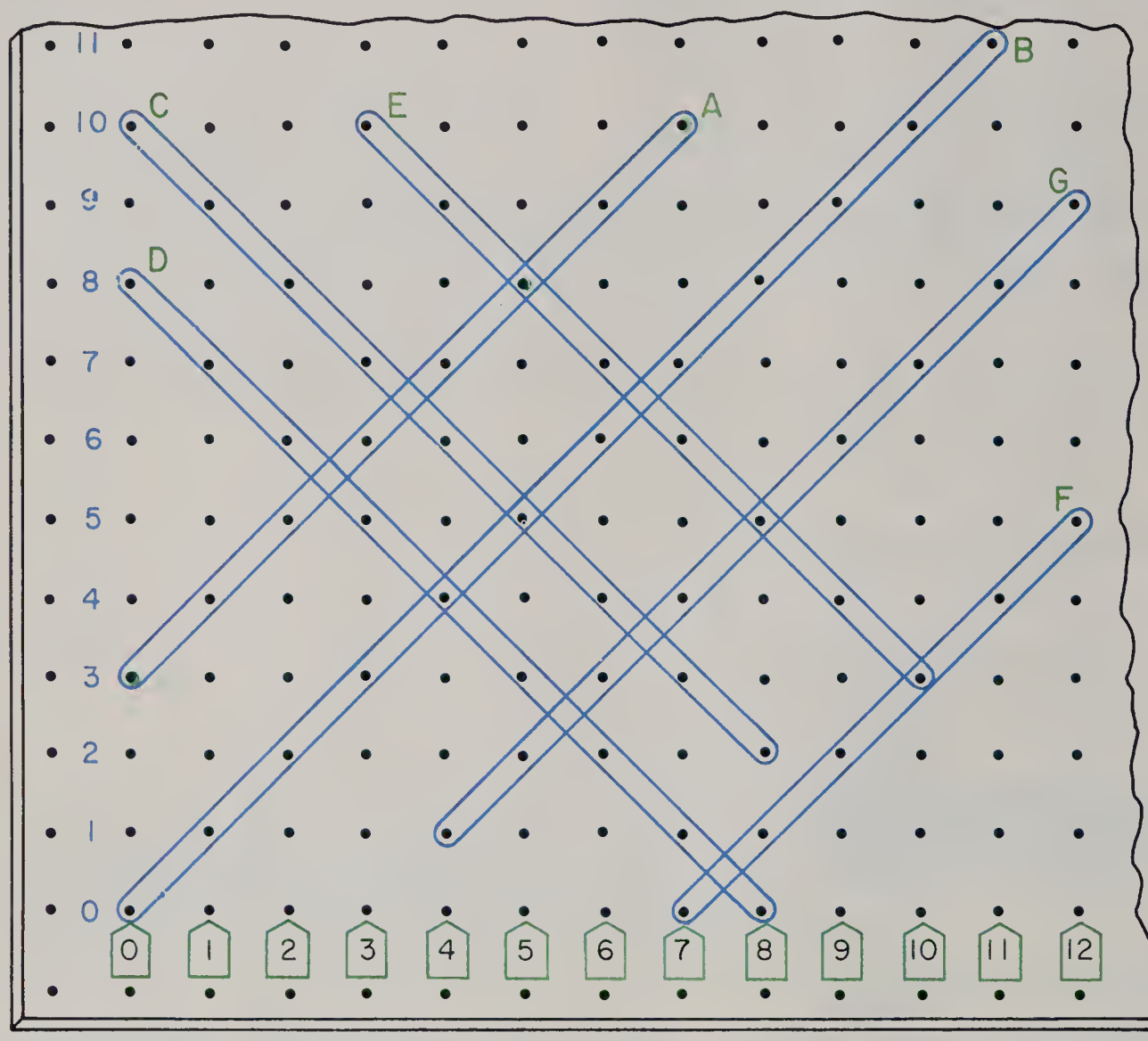
Can you guess what pattern the following two rules will lead to?

⌂ → ⌂ − $2\frac{1}{2}$

⌂ → $3\frac{1}{2}$ − ⌂

Another way to play WHAT'S MY RULE?

"On this scoreboard," Bill explained, "I have arranged rows of pegs to indicate results of imaginary games played. I've put a rubber band around the results of each game. Each group is lettered. I've given the rule for Game A and listed a few results. Find all the rules used and list 7 sample results."



Rule A: ☐ → ☐ + 3

☐	☐ + 3
7	10
5	8
0	3

Rule B:

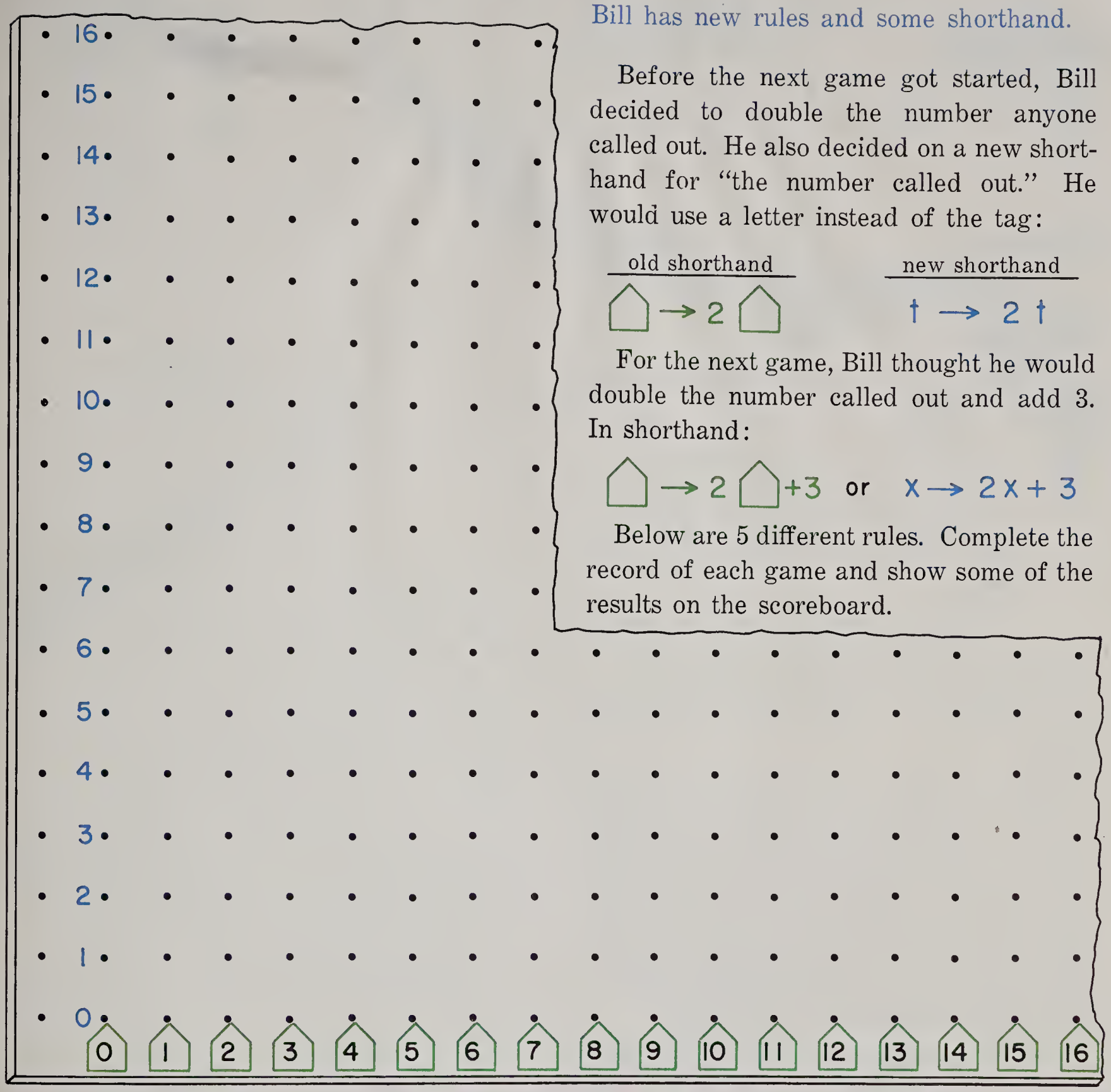
Rule C:

Rule D:

Rule E:

Rule F:

Rule G:



Bill has new rules and some shorthand.

Before the next game got started, Bill decided to double the number anyone called out. He also decided on a new shorthand for "the number called out." He would use a letter instead of the tag:

old shorthand	new shorthand
⌂ → 2 ⌂	t → 2 t

For the next game, Bill thought he would double the number called out and add 3. In shorthand:

⌂ → 2 ⌂ +3 or x → 2x + 3

Below are 5 different rules. Complete the record of each game and show some of the results on the scoreboard.

Bill's Rule J
t → 2 t

4	8

Bill's Rule K
x → 2x + 3

1	5

Bill's Rule L
s → 4x

2	8

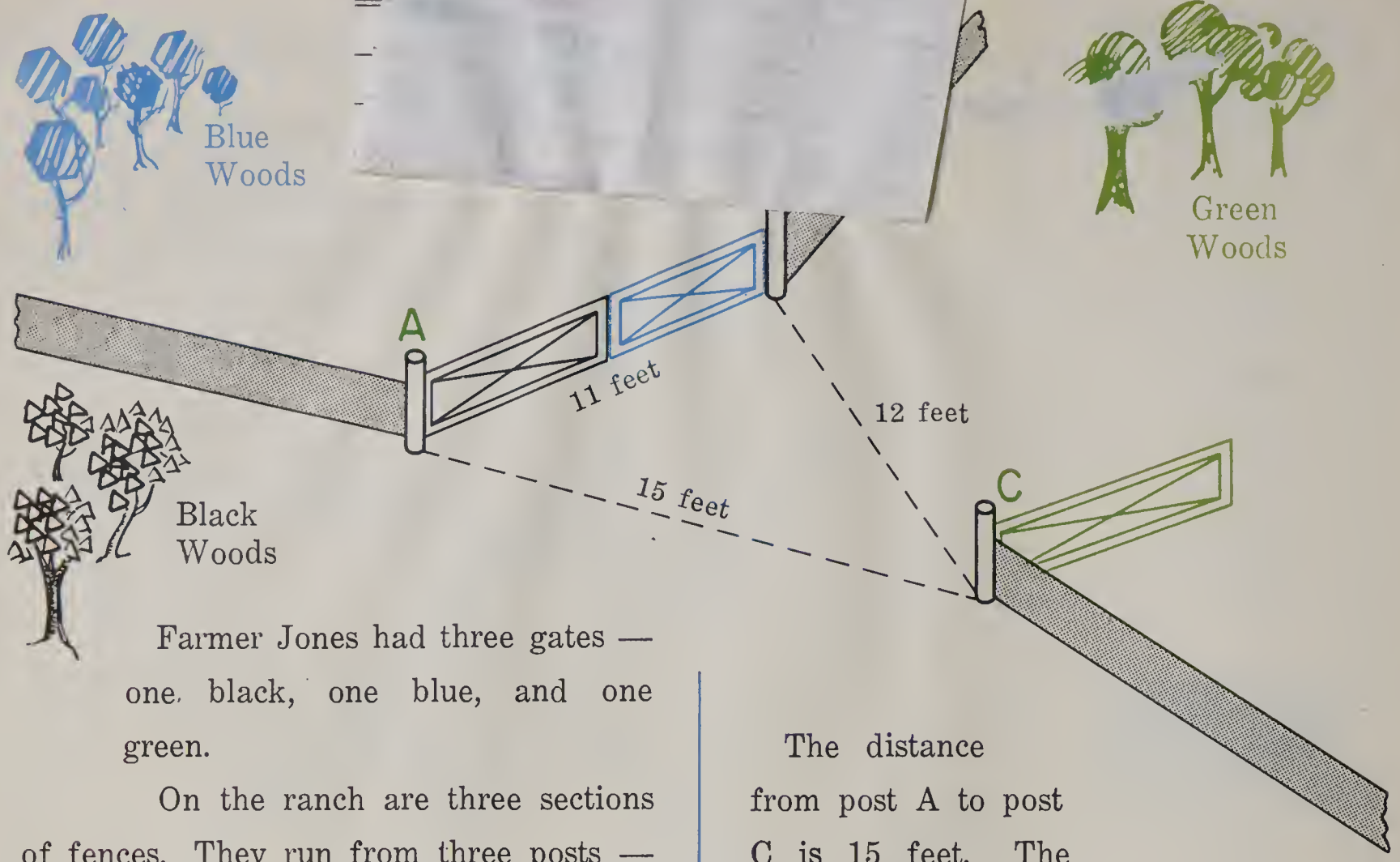
Bill's Rule M
r → r ÷ 2

6	3

Bill's Rule N
q → (q ÷ 2) − 3

8	1

Make up your own rules. Record game results on the scoreboard.



Farmer Jones had three gates — one black, one blue, and one green.

On the ranch are three sections of fences. They run from three posts — A, B, and C — out to the surrounding woods.

There is a bear that roams through the woods — the black woods, the blue woods, and the green woods. When the bear comes out of the blue woods, Farmer Jones closes the black and blue gates between posts A and B (as shown above).

If the bear comes out of the green woods, he closes the blue and the green fence between posts B and C.

If the bear comes out of the black woods, Farmer Jones closes the black and the green gates — between posts A and C.

In each case, Farmer Jones moves his cattle so the bear cannot get at them — there is always a fence or gates in between.

The distance from post A to post C is 15 feet. The distance from post A to post B is 11 feet. The distance from post B to post C is 12 feet.

Each time the gates are closed, they meet without any gap between them, and they do not overlap.

How long are the gates?

The black gate is ___ feet long.

The blue gate is ___ feet long.

The green gate is ___ feet long.

Take your time. You may find this very difficult. Think about it. Make a try. Then don't think about it for a while.

Come back to it another day. Try it again.

NOTE: If you discover the answer, you must not tell others. Each must discover it by himself!

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